DOCUMENT RESUME

ED 453 073 SE 064 733

AUTHOR	Puig, Luis, Ed.; Gutierrez, Angel, Ed.
TITLE	Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME 20) (20th, Valencia, Spain, July 8-12, 1996). Volume 4.
INSTITUTION	International Group for the Psychology of Mathematics Education.
ISSN	ISSN-0771-100X
PUB DATE	1996-00-00
NOTE	440p.; For volumes 1-4 and addenda, see SE 064 730-734.
PUB TYPE	Collected Works - Proceedings (021)
EDRS PRICE	MF01/PC18 Plus Postage.
DESCRIPTORS	*Educational Change; Educational Technology; Elementary Secondary Education; Higher Education; Knowledge Base for Teaching; *Language; Mathematical Concepts; *Mathematics Instruction; Problem Solving; Social Influences; Teacher Education

ABSTRACT

The fourth volume of this proceedings contains full research articles. Papers include: (1) "Conceptual and procedural approaches to problem-solving" (Y. Mohammad-Yusof and D. Tall); (2) "Teaching differential equations to chemistry and biology students: An overview on methodology of qualitative research. A case study" (M. Moreno and C. Azcarate); (3) "Language and assessment issues in mathematics education" (C. Morgan); (4) "Learning math in two languages" (J. Moschkovich); (5) "Hermeneutic experiences in constructing lessons and classroom research" (J.A. Mousley); (6) "Young students' informal knowledge of fractions" (H. Murray, A. Olivier, and P. Human); (7) "Tensions in the novice mathematician's induction to mathematical abstraction" (E. Nardi); (8) "Students assessment of an alternative approach to geometry" (L. Nasser, N. Sant'Anna, and A.P. Sant'Anna); (9) "Solving word problems with different mediators: How do deaf children perform?" (T. Nunes and C. Moreno); (10) "Telling definition and conditions: An ethnomethodological study of sociomathematical activity in classroom interaction" (M. Ohtani); (11) "Making sense of children's patterning" (J. Orton and A. Orton); (12) "Children's intuitive understanding of area measurement" (L. Outhred and M. Mitchelmore); (13) "Responsiveness: A key aspect of spatial problem solving" (K. Owens); (14) "The first algebraic learning. The failure of success" (M. Panizza, P. Sadowsky, and C. Sessa); (15) "Extending the educational conversation: Administrator's views of staff development" (T. Peluso, J.R. Becker, and B.J. Pence); (16) "The teaching of mathematics from within the school. Teachers and principals as researchers" (P. Perry, P. Gomezm, and P. Valero); (17) "Teachers' conceptions about mathematical assessment" (G. Philippou and C. Christou); (18) "Student teachers' conceptions of the rational number" (M. Pinto and D. Tall); (19) "Folding back to collect: Knowing you know what you need to know" (S. Pirie, L. Martin, and T. Kieren); (20) "Nouns, adjectives and images in elementary mathematics" (D. Pitta and E. Gray); (21) "Designing a domain for stochastic abstraction" (D. Pratt and R. Noss); (22) "Cord's story: An African American student points to the need for change in college mathematics pedagogy" (N.C. Presmeg); (23) "On dialectical relationships between signs and algebraic ideas" (L. Radford and M. Grenier); (24) "Exploiting understanding of data reduction" (C. Reading and J. Pegg); (25) "'Wouldn't it be good if we had a

Reproductions supplied by EDRS are the best that can be made
from the original document.

symbol to stand for any number': The relationship between natural language and symbolic notation in pattern description" (E. Redden); (26) "Enactivism as a methodology" (D.A. Reid); (27) "Teaching linear algebra: Role and nature of knowledges in logic and set theory which deal with some linear problems" (M. Rogalski); (28) "Ways of solving algebra problems: The influence of school culture" (T. Rojano, R. Sutherland, S. Ursini, S. Molyneux, and E. Jinich); (29) "On the introduction of real numbers in secondary school. An action-research experience" (I. Romero and L. Rico); (30) "Counting, estimation and the language of uncertainty" (T. Rowland); (31) "The transformation of mathematical objects in the didactic system: The case of the notion of function" (L. Ruiz Higueras and J.L. Rodriguez Fernandez); (32) "The use of illustrations in mathematics textbooks" (D. Santos-Bernard); (33) "Effects of computerized tools on prototypes of the function concept" (B.B. Schwarz and R. Hershkowitz); (34) "Affective constraints in mathematics education" (B. Southwell and M. Khamis); (35) "On the ability of high-school students to cope with a self-learning task in algebra" (J. Stupp and S. Vinner); (36) "Learning about teaching: The potential of specific mathematics teaching examples, presented on interactive multimedia" (P. Sullivan and J. Mousley); (37) "Mathematical modeling in the sciences through the eyes of Marina and Adam" (R. Sutherland, T. Rojano, S. Mochon, E. Jinch, and S. Molyneux); (38) "Pre-service teachers' problem solving strategies" (M. Taplin); (39) "Children's representation of the counting sequence 1-100: Cognitive structural development" (N. Thomas, J. Mulligan, and G.A. Goldin); (40) "College students' conceptions of variable" (M. Trigueros, S. Ursini, and A. Reyes); (41) "Seeing is reality: How graphic calculators may influence the conceptualization of limits" (L. Trouche and D. Guin); (42) "Children's misconceptions about the independence of random generators" (J.M. Truran); (43) "Children's use of a representativeness heuristic" (K.M. Truran); (44) "Two problems under one title: The case of division by zero" (P. Tsamir); (45) "Children's interaction and fraction learning in a computer microworld: Construction of the iterative fraction scheme" (R. Tzur); (46) "Precalculus and graphic calculators: The influence on teacher's beliefs" (P. Valero and C. Gomez); (47) "Conditional probability and the level of significance in the tests of hypotheses" (A. Vallecillos and C. Batanero); (48) "Students' awareness of the distributive property" (N. Vermeulen, A. Olivier and P. Human); (49) "Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in arithmetic word problem solving" (L. Verschaffel, E. De Corte, and I. Borghart); (50) "Semiotics as a descriptive framework in mathematical domains" (A. Vile and S. Lerman); (51) "Some psychological aspects of professional lives of secondary mathematics teachers--The humiliation, the frustration, the hope" (S. Vinner); (52) "Teachers' knowledge of pupils' errors in algebra" (E.K. Wanjala and A. Orton); (53) "Children's strategies for comparing two types of random generators" (J. Way); and (54) "Teaching to create discussion as mathematical argumentation" (T. Wood). (ASK)

Reproductions supplied by EDRS are the best that can be made
from the original document.

ED 453 073

Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education

ARITMÉTICA
PARA
LOS NIÑOS
94 x 6
36 + 45
17 ÷ 3
36 4
18
x 41

PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL HAS BEEN GRANTED BY

S. vanDenmolen

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☒ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

- Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

PME 20

July 8 - 12, 1996

University of Valencia
Valencia, Spain

Vol. 4

SE064453

BEST COPY AVAILABLE

Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education

Volume 4

International Group for the Psychology of
Mathematics Education

PME 20

Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education

Edited by

Luis Puig
Angel Gutiérrez

Volume 4

Valencia (Spain), July 8 - 12, 1996

Universitat de València
Dept. de Didàctica de la Matemàtica

4

Proceedings of the 20th Conference of the International
Group for the Psychology of Mathematics Education
Volume 4

Editors:
Luis Puig
Angel Gutiérrez

Dept. de Didàctica de la Matemàtica
Universitat de València
E.U. de Magisteri "Ausiàs March"
Apartado 22045
46071 - Valencia (Spain)
Fax: 34 - 6 - 3864487
E-mail: Luis.Puig@uv.es
Angel.Gutierrez@uv.es

Copyright © 1996 left to the Authors

All rights reserved

ISSN 0771-100X

Depósito Legal V-2190-1996

Cover: Reproduction of the cover of a Spanish textbook (Arithmetic for the children) written by Antonio Gallego Chaves and published by Saturnino Calleja in 1886.

Logo: The logo of the PME 20 Conference has been designed by Juan José Tornero, from the Dept. de Didàctica de l'Expressió Musical, Plàstica i Corporal of the Universitat de València.

Printed by Encuadernaciones Artesanas, S.L.
Periodista Badía 5; 46010 Valencia (Spain)

5

TABLE OF CONTENTS

VOLUME 4

Table of Contents 4-v

Research Reports (cont.)

Mohammad-Yusof, Y.; Tall, D. 4-3
Conceptual and procedural approaches to problem-solving
Moreno, M.; Azcárate, C. 4-11
Teaching differential equations to chemistry and biology students: An overview on methodology of qualitative research. A case study
Morgan, C. 4-19
Language and assessment issues in mathematics education
Moschkovich, J. 4-27
Learning math in two languages
Mousley, J.A. 4-35
Hermeneutic experiences in constructing lessons and classroom research
Murray, H.; Olivier, A.; Human, P. 4-43
Young students' informal knowledge of fractions
Nardi, E. 4-51
Tensions in the novice mathematician's induction to mathematical abstraction
Nasser, L.; Sant'Anna, N.; Sant'Anna, A.P. 4-59
Students assessment of an alternative approach to geometry
Nunes, T.; Moreno, C. 4-67
Solving word problems with different mediators: How do deaf children perform?
Ohtani, M. 4-75
Telling definition and conditions: An ethnomethodological study of sociomathematical activity in classroom interaction
Orton, J.; Orton, A. 4-83
Making sense of children's patterning
Outhred, L.; Mitchelmore, M. 4-91
Children's intuitive understanding of area measurement
Owens, K. 4-99
Responsiveness: A key aspect of spatial problem solving
Panizza, M.; Sadowsky, P.; Sessa, C. 4-107
The first algebraic learning. The failure of success
Peluso, T.; Becker, J.R.; Pence, B.J. 4-115
Extending the educational conversation: Administrator's views of staff development

Perry, P.; Gómez, P.; Valero, P. 4-123

The teaching of mathematics from within the school. Teachers and principals as researchers

Philippou, G.; Christou, C. 4-131

Teachers' conceptions about mathematical assessment

Pinto, M.; Tall, D. 4-139

Student teachers' conceptions of the rational number

Pirie, S.; Martin, L.; Kieren, T. 4-147

Folding back to collect: Knowing you know what you need to know

Pitta, D.; Gray, E. 4-155

Nouns, adjectives and images in elementary mathematics

Pratt, D.; Noss, R. 4-163

Designing a domain for stochastic abstraction

Presmeg, N.C. 4-171

Cord's story: An African American student points to the need for change in college mathematics pedagogy

Radford, L.; Grenier, M. 4-179

On dialectical relationships between signs and algebraic ideas

Reading, C.; Pegg, J. 4-187

Exploiting understanding of data reduction

Redden, E. 4-195

"Wouldn't it be good if we had a symbol to stand for any number": The relationship between natural language and symbolic notation in pattern description

Reid, D.A. 4-203

Enactivism as a methodology

Rogalski, M. 4-211

Teaching linear algebra: Role and nature of knowledges in logic and set theory which deal with some linear problems

Rojano, T.; Sutherland, R.; Ursini, S.; Molyneux, S.; Jinich, E. 4-219

Ways of solving algebra problems: The influence of school culture

Romero, I.; Rico, L. 4-227

On the introduction of real numbers in secondary school. An action-research experience

Rowland, T. 4-235

Counting, estimation and the language of uncertainty

Ruiz Higueras, L.; Rodríguez Fernández, J.L. 4-243

The transformation of mathematical objects in the didactic system: The case of the notion of function

Santos-Bernard, D. 4-251

The use of illustrations in mathematics textbooks

Schwarz, B.B.; Hershkowitz, R. 4-259

Effects of computerized tools on prototypes of the function concept

Southwell, B.; Khamis, M. 4-267
Affective constraints in mathematics education
Stupp, J.; Vinner, S. 4-275
On the ability of high-school students to cope with a self-learning task in algebra
Sullivan, P.; Mousley, J. 4-283
Learning about teaching: The potential of specific mathematics teaching examples, presented on interactive multimedia
Sutherland, R.; Rojano, T.; Mochon, S; Jinch E.; Molyneux, S. 4-291
Mathematical modelling in the sciences through the eyes of Marina and Adam
Taplin, M. 4-299
Pre-service teachers' problem solving strategies
Thomas, N.; Mulligan, J.; Goldin, G.A. 4-307
Children's representation of the counting sequence 1-100: Cognitive structural development
Trigueros, M.; Ursini, S.; Reyes, A. 4-315
College students' conceptions of variable
Trouche, L.; Guin, D. 4-323
Seeing is reality: How graphic calculators may influence the conceptualisation of limits
Truran, J.M. 4-331
Children's misconceptions about the independence of random generators
Truran, K.M. 4-339
Children's use of a representativeness heuristic
Tsamir, P. 4-347
Two problems under one title: The case of division by zero
Tzur, R. 4-355
Children's interaction and fraction learning in a computer microworld: Construction of the iterative fraction scheme
Valero, P.; Gómez, C. 4-363
Precalculus and graphic calculators: The influence on teacher's beliefs
Vallecillos, A.; Batanero, C. 4-371
Conditional probability and the level of significance in the tests of hypotheses
Vermeulen, N.; Olivier, A.; Human, P. 4-379
Students' awareness of the distributive property
Verschaffel, L.; De Corte, E.; Borghart, I. 4-387
Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in arithmetic word problem solving

Vile, A.; Lerman, S. 4-395
Semiotics as a descriptive framework in mathematical domains
Vinner, S. 4-403
Some psychological aspects of professional lives of secondary mathematics teachers - The humiliation, the frustration, the hope
Wanjala, E.K.; Orton, A. 4-411
Teachers' knowledge of pupils' errors in algebra
Way, J. 4-419
Children's strategies for comparing two types of random generators
Wood, T. 4-427
Teaching to create discussion as mathematical argumentation

RESEARCH REPORTS

(continuation)

Conceptual and Procedural Approaches to Problem-Solving

Yudariah Binte Mohammad Yusof
Department of Mathematics
Universiti Teknologi Malaysia
Locked Bag No 791, 80990 Johor Bahru
Malaysia

David Tall
Mathematics Education Research Centre
University of Warwick
Coventry CV4 7AL
United Kingdom

The introduction of problem-solving strategies has been shown to change students' attitudes to mathematics in ways that professors consider desirable. But does it change their overall strategies for doing mathematics? This paper reports data taken from students solving problems co-operatively who exhibited an overall improvement in attitudes (see Mohd Yusof & Tall, 1995). It indicates that some students who had said that "mathematics makes sense" approached problems in an open, creative way but that some lower attaining students who had stated that "mathematics does not make sense" treated problem-solving techniques as a new sequence of routine procedures.

There is a growing awareness that many university students are successful in learning how to carry out routine procedures to pass examinations,yet may not encounter experiences to encourage them to be creative and reflective. They are often given lectures that consist of theorems and proofs which do not encourage them to think mathematically. Problem-solving is seen as just a skill to be acquired. Studies have shown that the traditional approach is failing the majority of the students, not only the average students but more disturbingly also successful students. Students find great difficulties in constructing their own mathematical understanding (Davis & Vinner, 1986; Martin & Wheeler, 1987; Sierpinska, 1988; Eisenberg, 1991; Williams, 1991) and have a narrow view of the mathematics that shapes their mathematical behaviour (Schoenfeld, 1989; Vinner, 1994). Such difficulties were observed among Malaysian students (Mohd Yusof & Abd. Hamid, 1990; Razali & Tall, 1993). Nevertheless, research findings indicate that thinking mathematically or problem-solving can be taught with some success. For instance, Mason & Davis (1987) explored how people can develop their mathematical thinking, learning, and teaching by reflecting on their own experience. They argued that the technique of using meaningful vocabulary can help students to become more reflective and effective in mathematical learning. It was observed that students not only notice the use of the vocabulary and advice from tutors, but also remember it when the same language pattern (e.g. specialising, generalising, colloquial comments such as "What do I want?" etc.) was repeatedly used and their attention was explicitly drawn to it. Mohd Yusof & Tall (1995) reported that a course which provides students with experiences of sharing problem-solving activities has the effect of changing students' attitudes. Prior to the course the students generally regarded mathematics as abstract facts and procedures to be committed to memory, and had a range of negative attitudes such as fear of new problems, being unwilling to try new approaches, and giving up all too easily. After the course, students' attitudes changed in a positive direction. In this paper we investigate whether this change in attitude is accompanied by successful change in strategies for solving problems.

To investigate the manner in which students attack a problem, six groups were selected and given a problem which was relatively easy to state but did not have a straightforward algorithmic solution. The students taking part in the research were a mixture of third, fourth and fifth year undergraduates aged 18 to 21 in SSI (Industrial Science, majoring in Mathematics) and SPK (Computer Education), covering the full honours degree range. They had responded to a questionnaire in which they had been asked to indicate whether mathematics "makes sense" to them. Half the students agreed and half disagreed. Interestingly, the two groups had almost identical distributions of achievement in their previous year 's examination. (Table 1.)

	Degree Classification					
	I	II-1	II-2	III	P	F
Group *S*	3	11	7	1	0	0
Group *N*	3	13	5	1	0	0

Table 1 : Students for whom mathematics makes sense (Group *S*) and does not (Group *N*)

During the problem-solving course they had been encouraged to work in self-selected groups of three or four students. Several groups (by chance) consisted either entirely of students who declared that mathematics made sense (*S*) or that it did not (*N*). From these, three groups were chosen with all students in group *S* and three groups with students in group *N*. (Table 2.)

	Students	**Course**	**Degree Classification**	**Gender**	**Maths "makes sense"**
group 1 (S)	Sam	5 SPK	II–1	M	S
	Abel	4 SPK	II–2	M	S
	Henry	4 SPK	II–1	M	S
group 2 (S)	Sue	4 SPK	I	F	S
	Teresa	4 SPK	II–1	F	S
	Sasha	5 SPK	II–1	F	S
group 3 (S)	Rob	3 SSI	II–1	M	S
	Kline	3 SSI	II–1	M	S
	Ian	3 SSI	I	M	S
group 4 (N)	Hanna	5 SPK	II–1	F	N
	Katy	5 SPK	I	F	N
	Terry	5 SPK	I	M	N
group 5 (N)	Bob	5 SPK	II–2	M	N
	Yvonne	5 SPK	II–1	F	N
	Alma	4 SPK	II–1	F	N
	Pauline	5 SPK	II–2	F	N
group 6 (N)	Matt	5 SPK	II–1	M	N
	Al	4 SPK	II–2	M	N
	Holmes	5 SPK	III	M	N
	Ricky	5 SPK	II–2	M	N

Table 2 : The 6 groups of students selected for interview

Each group was invited at an appointed time for the session that lasted 40 minutes. The first 10 minutes served as a *relaxing* phase whereby the students were simply asked to talk about their mathematical experience at the university. For the next 30 minutes, they were given a problem to work on, as follows:

> A man lost on the Nullarbor Plain in Australia hears a train whistle due west of him. He cannot see the train but he knows that it runs on a very long, very straight track. His only chance to avoid perishing from thirst is to reach the track before the train has passed. Assuming that he and the train both travel at constant speeds, in which direction should he walk?
>
> *Mason, Burton & Stacey, 1982, p. 183.*

After being presented with the problem the students were left entirely on their own and their attempts in solving it were observed without any intervention. The interview then focused on the students' interpretation of their problem-solving experience.

During the course the students had been encouraged to view their activities in three phases – entry, attack and review, with appropriate activities for each (Mason et al, 1982). The purpose of the research was to see if the students used this structure as a framework for meaningful problem-solving.

The interview data provided some evidence of qualitatively different thinking between the various groups. For instance, the following excerpt from the beginning of the solution process when the students were in the "entry phase" indicates differences in mathematical understanding.

Students in group 1 spent a few moments establishing the meaning of "constant speed" and finally agreed it mean that both train and man were moving at different speeds.

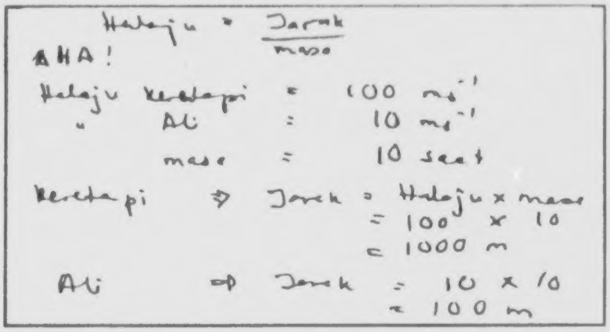
> ABEL: Constant speed …
> HENRY: The speed of the train must be the same.
> SAM: It is not the same.
> HENRY: Constant.
> SAM: Constant means it does not increase or decrease.
> ABEL: … the train travels say at 40 mph, Ali [the man] 4 mph. Ali will always travel at 4, the train always at 40. That is constant speed.
> SAM: I agree.
> HENRY: Hmm …
> ABEL: It is not the same speed but constant speed. Ali can be faster than the train …
> SAM: Ali and the train do not move the same, not at the same speed. But at their respective speeds … the same speed all the time. *group 1 (S)*

In contrast, group 6 students started from the misconception that constant speed meant that both man and train move at same speed. They quickly agreed with the meaning and no further reference was made to their interpretation of 'constant speed' until the end.

> MATT: … constant speed.
> AL: It means the same I think.
> HOLMES: Constant speed …, it's the same.
> MATT: Uniform ...
> AL: It means the man moves with the train at the same speed. Now OK … *group 6 (N)*

During the problem-solving, it could be seen that three of the six groups (the lower attaining *N* groups 5 and 6 and the younger *S* group 3) followed the techniques taught in the problem-solving course very rigidly. Of these three, the two *N* groups seemed to be doing it more religiously than the *S* group. They were more concerned to cover each phase in a sequence and could be seen to be working procedurally throughout. They interpreted the problem-solving technique as a procedure that they must follow step by step; it was as if they believed that precision in following each phase would guarantee them a solution. Most of their time was spent looking for formulas that could be used.

> PAULINE: We have already understood the question. We have introduced what we want, what we know. We have done that. OK now we can enter the attack [phase].
> YVONNE: What is the formula?
> BOB: Speed times time.
> YVONNE: The time is the same. The speed is …
> ALMA: We need to define speed first.
> BOB: I should remember how to do this. … Oh yes! speed is distance divided by time.
> PAULINE: Now the distance, we don't know how much, right? The distance between the man and the train.
> BOB: Let us assume the speed of the train is 100, the man 10. … …
> ALMA: OK we did some specialising …
> *group 5 (N)*



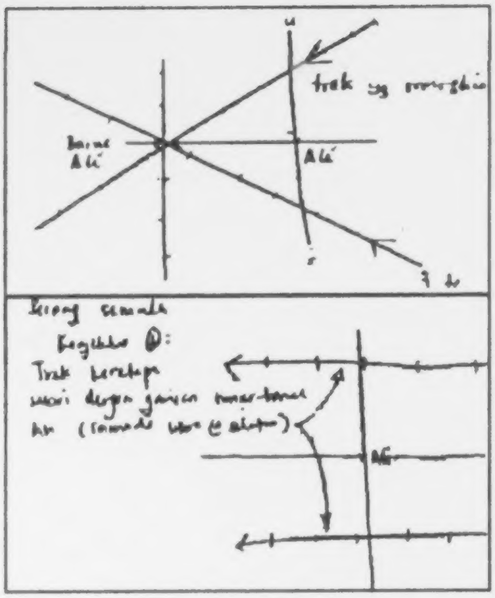
Halaju = $\frac{\text{Jarak}}{\text{masa}}$
AHA!
Halaju keretapi = 100 ms^{-1}
" Ali = 10 ms^{-1}
masa = 10 saat
Keretapi ⇒ Jarak = Halaju x masa
= 100 x 10
= 1000 m
Ali ⇒ Jarak = 10 x 10
= 100 m

> MATT: So, first we go to the entry phase.
> AL: OK. That is what we know. Now what we want is the direction in which the man should go.
> MATT: Anybody feel stuck or anything. The question is clear isn't it? … …
> RICKY: The concept of intersection. That is what we can say.
> HOLMES: The intersection point is the place the man has to go.
> MATT: OK, now we go to the attack phase.
> *group 6 (N)*

> ROB: We are stuck at this point.
> KLINE: Stuck. OK. write down we are stuck.
> IAN: Let's go back to what we want. What we want is the direction in which the man should walk. Direction, the man should go … west, east … …
> ROB: We are confident our assumption is correct so far. OK now we enter the attack phase.
> *group 3 (S)*

In none of these beginnings of solutions have the students thought in a broader conceptual fashion, for instance to consider the *direction* of the train, or to draw a diagram. The other groups were more involved in considering plausible ways to solve the problem by creating their own solution method.

> SAM: OK, so we conjecture that the train is moving towards the west. That is according to your understanding. But I have another suggestion. To me...we go back to entry phase OK? …
> ABEL: OK, I got it.



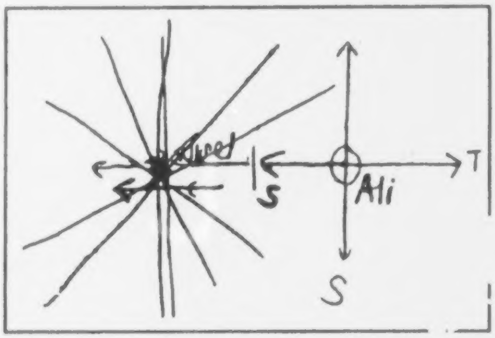
HENRY: No, no, no. Hang on. The train is moving to the west. ... But why should the man walk in this direction [pointing to the top diagram]? Why do you say that?

SAM: It is like this. Now this is just my idea. ... Say the man is here [drawing the next diagram] ... So he cannot go this way, otherwise he will be moving parallel to the track and may never reach the train.

HENRY: So according to what you say, the direction the man should walk is this one, to the north. OK, we can conjecture that.

ABEL: We have now answered the question. Now we want to justify whether it [the conjecture] is correct or not. *group 1 (S)*

In groups 2 (S) and 4 (N) one of the students thinks the train is moving west but others correct her and widen the issue:



TERESA: ...The train is moving to the west.

SASHA: Where does the train come from?

SUE: That is the problem. That is what we want to find out, it relates to the direction we want to go.

SASHA: Hmm ...We are stuck!

SUE If we know from where [the train is coming], we can find out where we want to go.

TERESA: Suppose we look at it this way. First say the man is here [pointing to a point on her paper]. Now we define where is his east, his west ... *group 2 (S)*

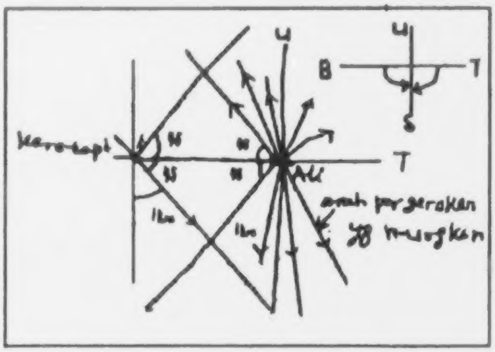
HANNA: We are wasting our time ... What I know, the question says, the train is moving towards the west. So the man must go towards the west as well.

TERRY: No! The question does not say the train is going west. But he heard [the whistle] due west of him.

KATY: Yeah, that is my understanding too. The man heard the train whistle due west of him. But this does not mean that the train is moving towards the west. We cannot come to that conclusion.

TERRY: How do we know from the whistle that the train is moving west or east... What is your reasoning? ... ...

TERRY: OK, that's it. So we conjecture that the man should walk to the north. I think we have a solution to the problem. But we are not finished yet, we need to justify this conjecture first. *group 4 (N)*



BEST COPY AVAILABLE

Four of the six groups (the three *S* groups 1, 2, 3 and the higher attaining *N* group 4) gave some evidence that they are able to carry out the mathematical processes to some extent. They show that they are capable of making judgements on the content and in making mathematical decisions for themselves. They also question the meaning of the task.

> The problem is very challenging. It does not require a specific formula or procedure that you have to apply to solve it. It is quite difficult. We got an idea what the answer is but to prove it is the hardest part. *group 1 (S)*

> We only managed to understand the question better towards the end of the discussion time. But I think we can solve the problem if we have more time. It is not difficult, but to generalise and to prove is very difficult ... We will keep on thinking about it until we get the answer. *group 2 (S)*

> The problems in the problem-solving course are interesting. Like this one. We have to think, work out what we want, what we do know before we actually work out what we don't know. ... The course is beneficial. It makes us sit down and see where to start. *group 3 (S)*

However, the other two *N* groups (5 and 6) have the notion that mathematical problems consist of direct application of facts and procedures. Their lower attainments on their examinations suggests they have less secure knowledge to bring to the solution process. Thus they are in an interesting position where they have built up their confidence to tackle problems and yet they find the problems very difficult.

> We tried to generate a few possible ideas. But we felt a bit put off because we couldn't recall the formulas. ... The problems are totally different from those in the maths course. In maths we always know what method to use. Here we have to find it out for ourselves. ... I think we have more confidence now. Before the [problem-solving] course we probably would have given up very easily. *group 5 (N)*

> We found it [the problem] very difficult. We are unsure of which formulas or methods to use. Even if we got a solution, we don't know whether our solution is right. ... Unlike problems in the problem-solving course, most of the problems in the maths course are simply applications of a ready rule. There is always a definite answer at the end. *group 6 (N)*

Discussion

Although none of the groups could provide a complete solution to the problem within the time limit, they were at least able to tackle the problem to make a start. *All* the student groups were very willing to tackle the problem without any overt sign of anxiety. Even though the problem remained unfinished, all three *S* groups and the higher attaining *N* group 4 considered that they could solve the problem given more time, (although based on their responses this may involve a lot more effort than they may have thought). Meanwhile, the other two *N* groups were seeking formulae appropriate for a solution and using the overall strategy of problem-solving as a procedure to attack the problem. Their response to problem-solving shows the same procedural format as their approach to traditional mathematics problems.

Byers & Erlwanger (1985) note that memory plays an important role in the understanding of mathematics. However, they suggest that it is *what* is remembered and *how* it is remembered that distinguishes those who understand from those who do not. Mathematical concepts are abstract entities requiring mental effort to construct relationships between the ideas. The students involved in this research have previously followed courses which place great demands on success in learning procedures and applying them to solve related problems so that they perceive mathematics more as a fixed body of knowledge to be learnt.

The problem-solving course has had various positive outcomes, for instance, the students have experienced the fact that not everything they do has to be immediately correct. If they were to fear making erroneous conjectures, the may not be able to solve any real problems. Although it is essential to get the right answer by the end of the process, it is evident that after the course, the students see that it is *how* they obtain an answer which is more important; making the intellectual journey to find the right methods and correct reasoning. It is possible to conjecture that the students' success in problem-solving during the course was sufficient to give them a sense of well being.

Although the students show little emotional reactions when solving an unexpected problem, opinions expressed in an attitudinal questionnaire suggest that group 2 have a positive attitude before and after the course. The majority of those in groups 1, 3 and 4 became more positively inclined during the course. Group 5 and 6's negative attitudes lessened during the course. The diminishing of fear and anxiety may be related to Skemp's (1979) idea of avoiding failure, and a perceived increase in confidence during the course involves seeing the task more as a goal to be achieved. In the case of all these students, there was a general sense of satisfaction expressed at the end of the problem-solving course. However, from the evidence of these investigations, it is clear that, for some, doing things procedurally is not an anti-goal for them as suggested by Skemp. To some of the students it is a goal, but it is a less suitable kind of goal.

Summary

The students involved in this research have long since learned that what matters most is to be able to carry out the procedures to do the mathematics. During the problem-solving course, although the majority of students showed that they are capable of carrying out the various processes of mathematical thinking and engage actively in problem-solving, the interviews emphasise that there are differences in the quality of the students' thinking. For instance for some lower attaining students for whom mathematics does not make sense, when faced with a problem appear to be more concerned about recalling and applying learned techniques to solve the problem rather than looking for insights, methods and reasons. Perhaps their contextual understanding of mathematical concepts is limited. Thus they lack confidence in carrying out the mathematical performance. Their reaction to the given mathematical problem gives an indication that they see problem-solving as just another procedure. While problem-solving, their emphasis is on applying learned techniques or ready rules to the task. They were using a procedural method and were not truly doing problem-solving. Their recorded discussion gave an indication of the way they do mathematics— in a procedural and non-conceptual way. After the problem-solving course, the tendency to lay emphasis on procedural aspects remains.

References

Byers, V. & Erlwanger, S., (1985). Memory in Mathematical Understanding. *Educational Studies in Mathematics, 16*, 259–281.

Davis, R. B. & Vinner, S., (1986). The Notion of Limit: Some Seemingly Unavoidable Misconception Stages. *Journal of Mathematical Behaviour*, 5, 281–303.

Eisenberg, T., (1991). Functions and Associated Learning Difficulties. In D. O. Tall (Ed.), *Advanced Mathematical Thinking.* Kluwer: Dordrecht..

Martin, W. G. & Wheeler, M. M., (1987). Infinity Concepts Among Preservice Elementary School Teachers. In J. C. Bergeron, N. Herscovics & Kieran, C., (Eds.), *Proceedings of PME 11*, III, 362–368. Montreal.

Mason, J. & Davis, J., (1987). The Use of Explicitly Introduced Vocabulary in Helping Students to Learn, and Teachers to Teach in Mathematics. In J. C. Bergeron, N. Herscovics & Kieran, C., (Eds.), *Proceedings of PME 11*, III, 275–281. Montreal.

Mason, J., Burton, L. & Stacey, K., (1982): *Thinking Mathematically.* London: Addison Wesley.

Mohd Yusof, Y. & Tall, D. O. (1995). Professors' Perceptions of Students' Mathematical Thinking: Do They Get What They Prefer or What They Expect? In L. Meira & D. Carraher, (Eds.), *Proceedings of PME* 19, II, 170–177, Recife.

Mohd Yusof, Y. & Abd Hamid, H., (1990). Teaching Students to Appreciate Mathematics. In P. K. Veloo, F. Lopez-Real & T. Singh (Eds.), *Proceedings of SEACME 5*, 328–334. Brunei.

Razali, M. R. & Tall, D. O., (1993). Diagnosing Students' Difficulties in Learning Mathematics. *International Journal of Mathematics Education, Science and Technology*, 24 (2), 209–220.

Schoenfeld, A. H., (1989). Explorations of Students' Mathematical Beliefs and Behavior. *Journal for Research in Mathematics Education*, 20 (4), 338–355.

Sierpinska, A., (1988). Epistemological Remarks on Functions. In A. Borbas (Ed.), *Proceedings of PME 12*, II, 568–573. Hungary.

Skemp, R. R., (1979). *Intelligence, Learning and Action*, London: Wiley.

Vinner, S. (1994): Traditional Mathematics Classrooms: Some Seemingly Unavoidable Features. In J. Pedro da Ponte & J. Filipe Matos (Eds.), *Proceedings of PME 18*, IV, 353–360. Lisbon.

Williams, S. R., (1991). Models of Limits Held by College Calculus Students. *Journal for Research in Mathematics Education*, 22 (3), 219–236.

TEACHING DIFFERENTIAL EQUATIONS TO CHEMISTRY AND BIOLOGY STUDENTS: AN OVERVIEW ON METHODOLOGY OF QUALITATIVE RESEARCH. A CASE STUDY.

Mar Moreno & Carmen Azcárate

Departamento de Didáctica de las Matemáticas y Ciencias Experimentales
Universidad Autónoma de Barcelona

ABSTRACT: *Intending to quest about the conceptions math teachers hold about how to teach Differential Equations to chemistry and biology students, we have devised a research tool which allows us to derive relevant information. We use different means to collect the adequate data related to the qualitative research, targeting the exploration of what teachers "say they do" and what "they do and would like to do". The use of concept maps and a questionnaire, along with a recorded interview, has revealed itself as an accurate means for the appropriate interpretation and analysis of data, as shown in the case study we hereby include.*

1 - Introduction

Concern about the teaching of advanced level mathematics and Artigue's (1989) studies on the learning of Differential Equations (D.E.) by students of Physics has motivated the interest in knowing and further exploring certain aspects of the teaching of mathematics in experimental schools, where both the discourse and the receiver of it merit a singular importance.

Our first suspicion was that mathematical materials taught during the initial undergraduate courses, did no differ substantially despite the various technical and scientific background of the students. Consequently, our first inquest on this field unveiled that no specific mathematical discourse existed for biologists and that the only variations, within the same mathematical contents taught in other schools, were based on the assessment level required. From this evidence, we established the research hereby (Moreno, 1995), which has allowed us to carry out an analysis and a deep reflection on some of the aspects of D.E. teaching in experimental science schools, keeping always in mind a math teacher's perspective.

D.E. make a segment of mathematics which has both historically and socially activated the interest among experts and profanes. History reminds us how problems in the area of Physics and Geometry acted as the propelling engine for D.E. to develop. Additionally, D.E. are highly interdisciplinary and their applicability outside the strict mathematical domains suggests interrogations of heterogeneous essence.

In view of what has so far been stated and taking into account the exploratory character of this study, we did no pre-establish any research hypothesis, although we actually selected some interest focus over which we wanted to articulate our scrutiny:

i) examine into the concept images educators have about teaching D.E. to experimental science students.

ii) discern and unveil different obstacles and difficulties for the "teaching transposition" from the "savoir savant" to the "savoir à enseigner" (Chevallard, 1991)

2 - Theoretical Framework

The theoretical basis for this study lays on certain mutually complementing cognition, teaching and pedagogical aspects, so as to build up an explanation model accounting for the observed, discerned and unveiled facts. So, we have given especial emphasis on the notion of "concept image" as developed by Tall and Vinner (1981), later re-elaborated and variegated by Vinner and Dreyfus (1989), Dreyfus (1990) and proved useful by Azcárate (1990) through her research on the notions of straight line gradient of derivative; we have assigned due relevance to the links between existent concept images and the representation and abstraction processes which underpin cognitive growth: Tall (1994, 1994a, 1994b, 1995), Dreyfus (1994) and Sfard (1991, 1994).

Other features have been taken into account which refer to teachers: attitudes and beliefs about the various elements involved in the education system, elements in decision-making, in planing and teaching style.

3 - Research methodology

The main interest of this report lies in this section. Considering the fact that the evidence from which we have reflected and drawn our conclusions comes from teachers' views and opinions, we decided to use various means for data gathering, in view to "validating" the information each of them would provide (Miles & Hubermann, 1987).

Participants in this study are four mathematicians, teaching staff in Mathematics Faculties who carry their teaching pursuit in chemistry or biology faculties of two different Spanish Universities. Because of their professional pathway and of their teaching demands, all of them have been teaching D.E. subjects for years, so they are fully and deeply acquainted with the topic.

The means for data gathering drawn up for all four sample teachers were:

- A concept map (C.M.) of the teachers conceptions about how to teach D.E. to chemistry and/or biology students.
- A questionnaire with four issues. Each issue had some open and closed questions about concepts and procedure viewpoints for the resolution of particular D.E.; additionally, some of those questions referred to various methodological facets

of their teaching. One of the questions in particular presented two different response solutions by two different students to a modeling problem; the teachers were asked to give their views on these two ways of solving the problem and about the concept and pedagogical aftermaths implied in them.

Additionally, various work materials provided by each teacher were available. All those work instruments were glued by means of a recorded interview, about one hour and a half long.

The purpose of the C.M. was to: gather information about the teachers' cognitive models on D.E. training, and try to grasp the significance and knowledge structure the teacher conveys to students. The purpose of the questionnaire was to: understand the teachers' view about particular concept and procedure aspects of the subject-matter, in itself, and as a complement to the concept map. The purpose of the interview was to: smooth up and clarify certain features both from the map and from the questionnaire which would not come up distinctly enough, otherwise risking an inaccurate interpretation of the gathered data.

4 - Analysis of research tools

4.1 - General analysis

Analyzing the data was done keeping the recorded interview as a reference point; thus, an analysis of the "concept map for every teacher" was carried on based on "their recorded explanations". In order to analyze the "questionnaire" we proceeded on "a question at a time" base and we relied on the recorded interview hoping that certain nuances not clear enough in the responses would burst up.

Finally, we tried to perform an "overall analysis" aiming at a consolidation of every teacher's features coming from all the sources available for us. This analysis ended up being very fruitful and plenty of particulars.

4.2 - Individual analysis

a) Concept map analysis

Keeping in mind Novak and Gowin (1988), we devised our own qualitative analysis tool. The analysis was performed at two levels: micro-analysis and macro-analysis.

Micro-analysis is a subtle and detailed one where semantic value of every straight and cross relationship contained in the C.M. was considered; additionally single words in the statements, degree of universality, semantic and syntactic congruence, kinds of links between words, etc., were considered as well.

Macro-analysis aimed at grouping terms which would encompass secondary concepts not carrying new information aside from that contained in source concepts. Evolving from the initial concept structure, this analysis allowed us to consider new and more general concept structures providing a more global view of every teacher's conceptions.

b) Questionnaire analysis.

Closed questions were scored from 1 to 5, translating those values into a qualitative scale such as:

__1____2____3____4____5__

LOW NORMAL HIGH

In order to ease the analysis of each item and in view of linking them to the specific targets of this study, we grouped them into three categories: different methods to solve D.E.; nature of submitted tasks; knowledge of and advantage taken from history.

Open questions allowed us to grasp the degree of flexibility and ability on the teachers' side in their search for clashing and interesting situations capable of provoking querying attitudes among the students.

Each question was analyzed at two separate descriptive levels: global and particular. At first level, participants' views were made explicit as far as the above mentioned categories were concerned. At the particular level, nuances of participants' answers upon items equally valued and which explanations reveal dissimilar motivations are made evident.

4.3 - Research general conclusions outlook

Analyzing the interviews was used as fulcrum for an accurate analysis of C.M.s and of questionnaires. Here is an outlook for some of the final conclusions:

- We describe three teaching styles: traditional, transitional and advanced, all showing math pursuits of some math teachers.
- Traditional and transitional styles focus training activities on process-like aspects, accumulating different techniques to solve D.E.; emphasis is given on training students to become competent D.E. solvers, leading to an incomplete development of D.E. "procept" (Tall, 1994a), to a faint flexibility of thought and to very poor conceptual designs.
- Advanced style sets out D.E. as "mathematical objects" and "foundation instruments" to formally conduct continuous deterministic models. It favors handling different representation approaches, so enlarging students' richness of their concept images linked to the concept and of a variety of mutual interconnections. All this maximizes cognitive retrieval and flexibilises proceptual view of D.E.

Next we summarize the analysis of one of the concept maps taken as a representative sample.

5 - A case study: concept map for teacher "A"

The following table summarizes the propositions appearing in the C.M. corresponding to teacher "A".

D.E.

1st LEVEL	ALGEBRA	ANALYSIS	APPLIED MATHEMATICS	
2nd LEVEL	MATHEMAT.	T.QUALITAT.	T.QUANTIT.	MODELISAT.
3rd LEVEL	EMPIRICAL KNOWLEDGE			EXPERIMENT. SCIENCES
4th LEVEL	PHILOSOPHY			NATURE

5.1 - Micro-analysis

Here are some of the aspects we highlight:

- No previously established hierarchical order is to be observed, but rather a series of ideas linked to the idea of D.E. It relates D.E. with other mathematical areas: analysis, algebra, applied mathematics and modelisation.
- It sets up a connection between "mode of mathematical thought" and "mode of empirical thought":

> [...] understanding mathematics is closely linked to applied mathematics. Therefore, it is linked to particular situations, and less to something one abstracts, i.e. what is called pure mathematics...[...] I think empiricism is one of the facets mathematical thought has. Let's say so, especially when it comes from within this descending chain: Mathematics, Empiricism and Philosophy.

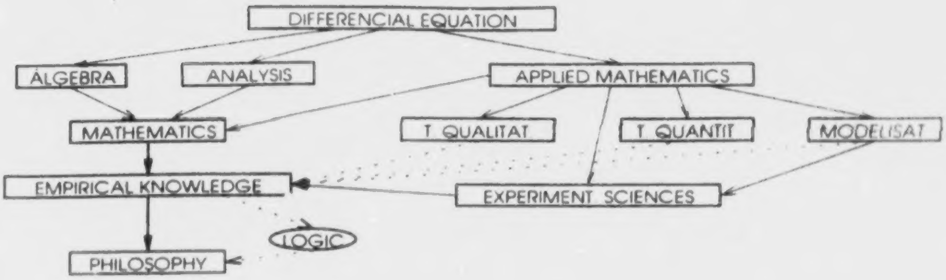
- The term modelisation leads him to analyze the importance of the specialist role who interprets the model and the difficulty in looking for the correct mathematical expressions.
- Interpreting the model, whether performed from a mathematical stand pint or from any other knowledge field, alters the teachers discourse to produce conflicting situations and interferences "not always appropriate or desirable" with a knowledge field diverse from our own. Assessment of these difficulties, added to the situation, environment, subject targets and the students' difficulties to assimilate all the information provided at a specific stage, influences the way the teacher acts, disposing away from his subject strategy the modelisation facets, increasingly focusing on a procedural approach to D.E.
- He sees the relationship between Mathematics and Experimental Sciences as very important:

> "A particular science progresses in as much as it mathematises itself. Even though this relationship is not set out in a straight forward manner, it is obvious it exists...[...] it conveys something which is abstract or ethereal in a more specific way".

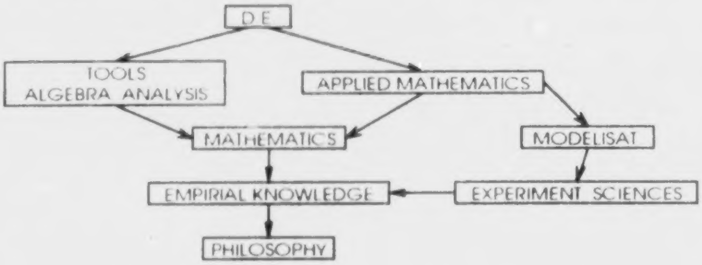
5.2 - Macro-analysis

There are two very neatly separated branches to be observed in the C.M.: i) one which unifies more particular concepts around Mathematics; ii) another one which diversifies originating from Applied Mathematics.

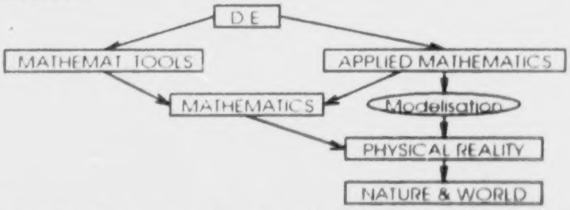
A first gaze at C.M. reveals:


DIFFERENCIAL EQUATION
ALGEBRA
ANALYSIS
APPLIED MATHEMATICS
MATHEMATICS
T QUALITAT
T QUANTIT
MODELISAT
EMPIRICAL KNOWLEDGE
EXPERIMENT SCIENCES
LOGIC
PHILOSOPHY

A first zoom over C.M. will allow us to unify "concepts" and reach a C.M. with a central backbone axis. On the one hand, the cross connection between Applied Mathematics and Mathematics; and on the other, that of Algebra and Analysis with Mathematics, both granting us to consider a unique hierarchical level which includes Algebra, Analysis and Applied Mathematics, converging to the term Mathematics. From there on, the ascending chain keeps on including Experimental Sciences, among other components.

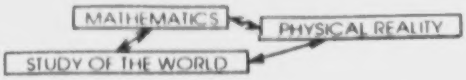

D E
TOOLS
ALGEBRA ANALYSIS
APPLIED MATHEMATICS
MATHEMATICS
MODELISAT
EMPIRIAL KNOWLEDGE
EXPERIMENT SCIENCES
PHILOSOPHY

Based on the relationship established between empirical knowledge and Experimental Sciences, we unify both terms into a unique term which reads "study of physical reality". Furthermore, we encompass the term Philosophy into a term which reads: "Nature and World", so that the C.M. would now display the following appearance:


D E
MATHEMAT TOOLS
APPLIED MATHEMATICS
MATHEMATICS
Modelisation
PHYSICAL REALITY
NATURE & WORLD

Contained in this third approach, the main concepts are: Differential Equation; Pure Mathematics; Applied Mathematics; Physical world reality; Study of the world and of nature.

Zooming once more, we perceive that the three main ideas underlying the whole thing are: MATHEMATICS / PHYSICAL REALITY / STUDY OF THE WORLD. His conception is fairly close to that of Newton and Leibniz, and to that of some sixteenth century mathematicians who tried to compound two very important aspects: Mathematics and Physical world reality. This way, the C.M. becomes:



A deep reflection will make us aware that all the initial ideas in the initial C.M. are hereby represented.

5.3 - Conclusions from teacher "A"'s conceptions

- The global and universal pattern of teacher "A"'s C.M. conforms with his personal interest and curiosity about Sciences and Mathematics; this is a circumstance which is not matched with any corresponding direct transposition to everyday teaching practice
- The global and universal pattern observed for this teacher may cut both ways and might worm against him. On one side, it allows him for more flexibility and gives him an overall view of Mathematics within Experimental Sciences; on the other, he is led to enhance enciclopedism and to present mathematics a an "object of interest in, and by, itself" independent from other knowledge areas.
- Treatment of modelisation becomes a friction knob between what he "thinks" and what he "does". Importance given to this aspect of D.E. in the C.M., does not match its later treatment in the classroom.
- This teacher's teaching practice shrinks down to a summation of algorithms and solving techniques to approach particular kinds of D.E.

6 - Methodological conclusions

- Richness of C.M.s, along with each teacher's recorded explanations, have provided such a variety of details and information that we have been able to analyze and interpret each participant's ideas at a satisfactory level of accuracy and objectiveness.
- Both levels of analysis of C.M.s has been very valuable. First, micro-analysis has allowed for a fully detailed knowledge of participants' conceptions on D.E. teaching, and to access to very specific details, which would have remained hardly accessible otherwise. At the same time, macro-analysis has provided with very general ideas about basic aspects of the teaching and learning of mathematics, to which all teachers pay especial attention.
- Despite difficulties in analyzing and coding the data, the methodology especially designed for this research is highly valued, endorsing its use as a tool for gathering and analyzing data related to the under way research.

BIBLIOGRAPHY:

Artigue, M., (1989). "Une recherche d'ingénierie didactique sur l'enseignement des equations différentielles en primer cycle universitaire", *Cahiers du Séminaire de Didactique des Mathématiques et de l'Informatique de Grenoble*. Grenoble: IMAG.

Azcárate, C. (1990). *La velocidad: introducción al concepto de derivada*.. Tesis de doctorado. Universidad Autónoma de Barcelona.

Chevallard, Y. (1991). *La transposition didactique. Du savoir savant au savoir enseigné. Un exemple de la transposition didactique*. Grenoble: La Pensée Sauvage.

Dreyfus, T. (1990). Mathematics and Cognition. Advanced Mathematical Thinking. En Nesher & Kilpatrick (Eds.). *ICMI Study Series*. Cambridge University Press, pp. 113-134.

Miles, M.B. & Huberman, A.M. (1987). *Data Analysis. A Source of New Methods*. Beverly Hills: Sage Publications.

Moreno, M.M. (1995). *Enseñanza de las Ecuaciones Diferenciales a químicos y biólogos desde la perspectiva del profesor de matemáticas. Estudio de casos*. Tesis de maestría, programa de Doctorado del Departamento de Didáctica de la Matemática y de las Ciencias Experimentales de la Universidad Autónoma de Barcelona.

Novak, J.D. & Gowin, D.B. (1988). *Aprendiendo a aprender*. Madrid: Martínez Roca.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, Vol. 22, pp. 1-36.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, Vol. 14. No 1, pp. 44-54.

Tall, D.O. & Vinner, S. (1981). Concept image and concept definition in Mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, No 12, pp. 151-169.

Tall, D.O. (1994). The Psychology of Advanced Mathematical Thinking. En Tall, D. (Ed.), *Advanced Mathematical Thinking*, pp. 4-25. Dordrecht/ Boston/ London: Kluwer.

Tall, D.O. (1994a). A Versatile Theory of Visualisation and Symbolisation in Mathematics. *Plenary Presentation at the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques*, Toulouse, Francia, julio 1994, pp. 1-12.

Tall, D.O. (1994b). Understanding the Processes of Advanced Mathematical Thinking. *An invited ICMI lecture at the International Congress of Mathematicians*, Zurich, agosto 1994, pp. 1-9.

Tall, D.O. (1995). Cognitive Growth in Elementary and Advanced Mathematical Thinking. *Plenary Lecture at the Annual Conference of the International Group for the Psychology of Mathematics Education*, Recife, Brasil, julio 1995, pp. 1-9.

Vinner, S. (1989). The avoidance of visual considerations in calculus students. *Focus on Learning Problems in Mathematics*, Vol. 11, No 2, pp. 149-155.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, Vol. 20, pp. 356-366.

LANGUAGE AND ASSESSMENT ISSUES IN MATHEMATICS EDUCATION

Candia Morgan

Institute of Education, University of London

Recent international concern with assessment is largely based on the assumption that close scrutiny of texts produced by students in the classroom context will provide valid evidence of their mathematical thinking and attainment. There are, however, contradictions between such an assumption and constructivist epistemologies. In this paper, it is argued that it cannot be assumed that a student's linguistic production transparently represents his or her mathematical thinking. Examples are provided, supporting this conclusion, of readings of student texts produced by experienced teachers. The implications for teachers and students, for thinking about assessment, and for research are discussed.

There has recently been considerable international concern with the development of new modes of assessment in mathematics (e.g. Houston, 1993; Lesh & Lamon, 1992; Niss, 1993). In particular, 'performance' or 'authentic' assessment is being discussed: that is, assessment which actually attempts to assess the learning that takes place during everyday classroom activity, often involving teachers directly in the assessment process. However, in spite of increased teacher involvement in both everyday and 'high stakes' assessment (for example, in the UK, in teacher assessment of the National Curriculum and in assessment of 'coursework' as part of the GCSE examination at 16+), very little detailed research, either in mathematics education or in other areas of the curriculum, has considered teachers' assessment practices (Torrance, 1995). In this paper, assumptions underlying these developments in assessment are questioned, drawing on analyses of teachers' practices in the particular context of the assessment of reports of investigative work in mathematics (Morgan, 1996).

Most of the evidence available to teachers for assessment purposes takes the form of linguistic, symbolic and graphic texts produced by students. These texts may be in oral form (gathered, for example, from formal interviews, incidental conversation between student and teacher, or overhearing of student-student conversation in the classroom) or in written form. In practice it seems likely that, particularly in 'high stakes' assessment situations in which teachers are concerned to be able to validate their professional judgements by providing evidence to colleagues or to external bodies, written texts will play the more important role. Much of the discussion that follows and the examples illustrating it specifically concern written texts; I would suggest, however, that the arguments are, on the whole, equally applicable to the assessment of oral texts.

Although it is claimed by many of those advocating the greater use of writing as a way of learning in the mathematics classroom that students' writing provides the teacher with insight into student thinking (e.g. Borasi & Rose, 1989; Miller, 1992), it is simultaneously widely acknowledged that many students do not have the linguistic skills or judgement necessary to represent their thinking adequately in written form and that there may be a mismatch between assessments formed solely on the basis of written work and those which take other sources into account (MacNamara & Roper, 1992; NCTM, 1995). This acknowledgement of problems in taking written texts as evidence of thinking, however, locates the responsibility for any shortcomings with the students and in the text itself: if the student's language skills or judgement about what to include in the text were better then the written evidence could be taken as unproblematic. It is thus assumed that there exists a notional 'perfect' text that would provide the teacher with a transparent representation of the student's intended meanings.

Such an assumption is, however, based on a 'common sense' or naive transmission view of the nature of communication. While mathematics educators have widely accepted some version of a constructivist epistemology in relation to the ways in which children make sense of their experiences, including the verbal and non-verbal texts available to them, thinking on assessment and, indeed, much research methodology still tends to work within a traditional paradigm in which meaning resides within the text, independent of the reader, carrying the author's intentions exactly. The assessor's or researcher's role is thus to 'extract the meaning' from the text. A more consistent epistemology, however, would suggest that there is no necessary simple correspondence between a piece of text and the meanings its various readers construct. Rather, the meanings constructed will depend on the resources brought to bear on the text by individual readers. These resources will vary according to the discourse within which the text is read and the positions adopted by a particular reader within that discourse as well as the reader's previous experience. As Kress (1989) argues, the text itself constructs an "ideal reader" by providing a reading position from which the text is unproblematic and "natural", but readers do not necessarily take up the "ideal" position and may resist the text by interpreting it within a different discourse using a different set of resources. If the student writer is to convey her intentions most effectively to her teacher-reader it is necessary for her to share a knowledge of the teacher's resources and most likely reading position. It seems, however, that some mathematics students do not share the textual preferences of their teacher-readers and may thus produce texts constructing ideal readers which do not match the teachers' expectations and reading positions (Guillerault & Laborde, 1982; Morgan, 1992). When such mismatch occurs, the teacher, acting within a discourse of school mathematics in which she is an authority (and hence entitled to define what is acceptable within the discourse), is likely to interpret the failure of communication as a failure on the part of the student either to communicate effectively or to understand the mathematical subject matter in the desired way.

Variations in the form of a text have an effect on readers' evaluations of its content (e.g. Hake & Williams, 1981; Anderson, 1988; Wade & Wood, 1979) and of the intelligence, understanding or other personal characteristics of the author (Kress, 1990; Hayes et al., 1992) although the nature of the effect will vary between different contexts. Kress (1990) exemplifies this through his analysis of two students' economics essays awarded very different marks by their teacher. While both cover the same areas of 'content' without error, the language of one shows less control over conventional forms of academic argument and is thus assessed to show less control of the subject matter. There is, of course, some difficulty in separating form from content in this way as it might be argued that the formation of an argument acceptable to a particular discipline forms part of the 'content' of that discipline; moreover, the exact nature of 'content' is likely to be affected by the form in which it is presented. Nevertheless, it is not possible to discern from the textual evidence alone whether the perceived weaknesses in this text arise from a lack of understanding of the subject matter or from a lack of awareness of the expectations of this genre of writing. There is a need to examine the relationships between forms of language used by students, their forms of understanding, and the assessments made by teachers on the basis of linguistic evidence.

In the domain of mathematics education, in spite of interest in developing more diverse and valid means of assessment, little attention has been paid to mathematics teachers' practices in forming evaluations of students' written work, perhaps because of a dominant traditional assumption that answers in mathematics are unambiguously right or wrong and that evaluation is thus unproblematic. One exception, looking at students' work in a traditional style of 'problem solving' in the US, is a study by Flener & Reedy (1990) who found that some teachers were unwilling to accept answers that were expressed in unconventional forms. New developments in assessment which involve more open problems have not, on the whole, addressed the issue of how the more diverse responses likely to be produced by students may be received by teachers (Collis, 1992).

The examples discussed in the remainder of this paper illustrate two fundamental issues related to the use of linguistic evidence of mathematical understanding: the effects on a teacher-reader of student choices of language in constructing their texts and the variety that may exist between different teachers' readings of the same text. The extracts are taken from interviews with experienced mathematics teachers during which they talked aloud as they read and assessed a number of pieces of student work on investigative tasks. Further details of the study from which these are taken are given elsewhere (Morgan, 1994; 1996); my intention in offering these examples here is to illustrate theoretical issues related to language and assessment rather than to present the results of empirical research.

Two examples

Example 1: Judgement of intellectual competence on the basis of linguistic 'style'

Among the texts read by the teachers were two extracts containing valid general solutions to the same problem (finding a relationship between the dimensions of a trapezium drawn on isometric paper and the number of unit triangles contained within it):

Student No.2:

If you add the top length and the bottom length, then multiply by the slant length, you get the number of unit triangles.

For example:

$3 + 5 = 8$ *and* $2 + 4 = 6$
$8 \times 2 = \underline{16}$ $6 \times 2 = \underline{12}$

This, therefore is the formula:

(TOP LENGTH + BOTTOM LENGTH) × SLANT LENGTH = No. OF TRIANGLES

Student No.3:

If you add together both the top length and the bottom length and times it by the slant length, you will end up with the number of unit triangles in that trapezium.

You can write this as $S(T + B)$

(Both extracts were typed in order to avoid judgements based on handwriting.)

The comparable parts of the two texts, as one teacher, Dan, remarked, are very similar. However, Dan added:

> *Number 2 gives me the impression they obviously know what they're talking about whereas this one [No.3], although it says almost exactly the same thing in different words, er, it doesn't give me the same impression.*

Obvious differences between the two texts include the fact that No.2 gives two examples and has used verbal variable names, while No.3 has used algebraic symbols for her formula. Dan had commented on these differences earlier, claiming that they did not greatly affect his assessment of the students (see Morgan, 1994). His "impression" appears to be based rather on the verbal descriptions of the procedure which, as he says, appear very similar with only slight variations in form. An analysis of the differences between the language used in the two texts suggests a number of aspects which may have affected Dan's reading:

- The use by No.3 of *times* rather than *multiply* is less formally 'mathematical' and may be read as a remnant of the early years of mathematics schooling and hence as a sign of immaturity.
- No.3's procedure is more 'wordy' using, for example, *add together both* rather than simply *add*. The number of unit triangles is also qualified as being *in that trapezium*. These additional words include reference to the concrete lengths, numbers or shapes. The procedure may thus be read as being at a lower level of abstraction.
- The use of *you will end up with* rather than *you get*, by using the future tense, also suggests a more concrete procedure, located in time (Kress, 1989).
- The introduction of the final formula by *You can write this as . . .* presents the symbolic formula merely as an alternative to the verbal procedure. No.2's announcement *This therefore is the formula*, on the other hand, displays the formula as a product in its own right which follows logically from the procedure rather than merely being equivalent to it. This may be read as an indication that No.2 has a better understanding of the importance of the relational formula and the fundamental difference between this and the verbal procedure, even though she has not used algebraic symbols to express it. The contrasting modality of these two statements also suggests that the two students differ in their levels of confidence.

While it is not possible to say which of these features specifically contributed to Dan's impression of No.3's lesser understanding, there is clearly a mismatch between the student's text and Dan's expectations which appears to have affected his evaluation of the whole of the student's performance. This analysis of possible sources for Dan's different evaluations of the two extracts points to the subtle nature of the relationship between the linguistic form of the text produced by the student and the evaluation of her general intellectual 'ability'. Significantly, Dan himself was unable to identify the features of the text which gave rise to his impressions.

Example 2: Different readings of the same text

My second example illustrates the lack of uniformity in various teacher-reader responses to a single text. Working on a different problem, Steven derived a correct algebraic generalisation, $(A + A) + \left(\frac{A}{2}\right) = b$, from empirically gathered data and applied it to a specific example too large to be checked by experiment. He then presented an alternative method of achieving the same answer by applying the formula to a smaller value and multiplying by a scale factor:

An alternative way to do this would be to take the result of a pile starting at 10 and multiply it by 10

$$(10+10)=20 \qquad \left(\frac{10}{2}\right)=5$$

e.g.

$$20+5=25$$

$$25\times 10=250$$

No further justification of this method is provided in the text. In particular, there is no indication of how it was derived. The scaling process is, however, completely valid given the directly proportional relationship between the variables in this situation. Nevertheless, it appeared to surprise the teachers reading this text and a number of different, and even contradictory, readings and evaluations of the student's understanding and competence were made.

Charles:

> *Um ok so I mean he's found the rule and he's quite successfully used it from what I can see to make predictions about what's going to happen for things that he obviously can't set up. So that shows that he understands the formula which he's come up with quite well, I think. There's also found some sort of linearity in the results whereby he can just multiply up numbers which again shows quite a good understanding of the problem I think.*

Charles, having judged Steven to understand the original formula, recognises the mathematical validity of the alternative method and takes this as a sign of the student's "good understanding".

Grant:

> *It's interesting that the next part works, I don't know if it works for everything or it just works for this but he's spotted it and again he hasn't really looked into it any further. He's done it for one case but whether it would work for any other case is er I don't know, he hasn't looked into it. . . And he's used it in the next part er used the this multiplying section in the next part and it's just a knowledge of number that's got him there I think intuition whatever. He may have guessed at a few and found one that works for it*

Grant himself expresses uncertainty about whether the method would work in general. Perhaps because of this uncertainty, his narrative explaining how Steven might have arrived at the method devalues the student's achievement, suggesting that the processes involved were not really 'mathematical': "spotting" the method, not looking into it properly, guessing, using "just a knowledge of number" or "intuition". Steven is clearly not being given credit either for the result itself or for the processes he may have gone through in order to arrive at it.

Harry:

> *and he's got another formula here . . . I don't really understand what he's done here . . . So he's produced another formula where . . . he's taken the result of a pile starting at ten and multiplying by ten and I don't understand what he's done there . . . I would have asked him to explain a bit further. He's - the initial formula with two hundred and fifty is proved to be correct and he's trying to extend it, he's trying to look for other ways, maybe he has realised that two hundred and fifty could be the exact answer or maybe not. So he's trying other ways to explain some of the inconsistencies that he's seen but I think greater explanation needed here.*

Like Grant, Harry seems to have some difficulty in making sense of this method and does not appear to recognise the equivalence between the original formula and the alternative method. In spite of this, he is able to construct yet another narrative to explain the student's intentions, stressing by repetition the suggestion that Steven has been "trying" (possibly with the implication that he has not succeeded). Harry locates the responsibility for his own failure to understand in the inadequacies of the student's text.

The differences between the various readings lie not only in different interpretations of the mathematical content of the text but in different interpretations of the student's level of understanding and different hypotheses about the methods that the student might have used in order to achieve his results. There may be a connection between these two aspects; it is Charles, expressing the clearest understanding of the relationship of the alternative method to the linearity of the situation, who makes the most positive evaluation of Steven's understanding, while Grant and Harry, apparently uncertain of the general validity of the method, both construct pictures of the student working in relatively unstructured or experimental ways.

It could be argued that this example involved a non-standard result and that the demand for greater explanation (both of the result itself and of the processes gone through in arriving at it) is therefore fully justified. This, however, begs the question of how the student is to know which of his results are non-standard or are likely to be perceived by the teacher-assessor as non-standard and hence in need of greater explanation. Moreover, there is no guarantee that further additions to the text would lead to greater conformity in teachers' readings of it.

Implications

It is clear from the examples discussed above that there is no simple one-to-one relationship between the text produced by a student and a teacher's assessment of the student's mathematical thinking on the basis of reading the text. The readings produced appear to be influenced by the individual teachers' expectations about the probable nature of such a text as well as by the teachers' own understandings of the mathematics involved. It must thus be asked what is being assessed: is it the student's mathematical understanding or competence or is it his or her competence in creating a text that will be judged to be appropriate within the genre anticipated by the teacher? In spite of wide spread awareness of the difficulties that some aspects of mathematical language may cause for many learners as readers and listeners, far less attention has been given within mathematics education to the ways in which students may learn to produce mathematical language themselves, particularly written language. While 'Writing to Learn Mathematics' has many advocates, few have addressed 'Learning to Write Mathematics' except in the context of relatively limited (though important) and generally short and formulaic types of text such as symbolic generalisations or formal proofs.

Innovations in 'authentic' assessment often involve students in producing a variety of more extended texts, including in particular reports and explanations. Subtle differences in the language used may lead, as in the first example above, to differences in the "impression" of a student's level of understanding achieved by a teacher-assessor. It seems likely that most teachers of mathematics (and indeed of other curriculum subjects (Langer & Applebee, 1987)) do not have the explicit knowledge of forms of language necessary either to diagnose the ways in which their own judgements are influenced by various styles of student writing or to provide adequate guidance to their students on how to produce texts that are likely to create more positive impressions. The analysis in example 1 provides some suggestions of linguistic features that may be influential; others related to the genre of reports of investigative work are discussed in Morgan (1996). There is a need, however, for further research to investigate the forms of language that are likely to be valued in various genres of mathematical text expected of students and to make this knowledge accessible to teachers and students.

In recommending that teachers and students should pay attention to developing mathematical language it is important to bear in mind that I am not suggesting that a better grasp of language will necessarily lead to the production by students of texts that will bring teachers to construct more accurate representations of the students' thinking and their mathematical activity; no text is entirely transparent. What knowledge about language is more likely to achieve is the production by students of texts that, by matching the teacher-assessor's expectations about forms of words, syntax and organisation, will be more likely to be evaluated positively. At present some students, primarily those from more privileged backgrounds, already possess adequate linguistic awareness to achieve this; explicit attention to language in the mathematics classroom could enable more students to participate in the discourse on an equal basis.

Although pointing to the difficulties and contradictions involved in modes of assessment that may be labelled 'authentic', I am certainly not advocating a resort to so-called 'objective' testing. Here too there are substantial problems with validity and, perhaps most significantly, conflicts with curriculum objectives; these have been adequately critiqued elsewhere. Nevertheless, while 'authentic' assessment may be preferable because of its less objectionable effects on the mathematics curriculum, it cannot be assumed that the assessments of student understanding achieved will be any more valid than those achieved by more traditional means. As well as raising questions about the validity of assessment of student understanding based on the evidence of linguistic texts, an acknowledgement that language does not transparently transmit an author's intentions and that different readers may construct different meanings from the same text must also raise questions for researchers making use of textual data (both written and oral). Again, in raising this issue I am certainly not advocating abandoning qualitative, interpretive methodologies. I am, however, suggesting that a knowledge of the ways in which different forms of language within a text may influence the inferences made by the reader-researcher together with conscious attention to such forms during the analytic process would be likely not only to enrich the resulting analyses but also to avoid distortions resulting from a student-subject's lack of control of conventional forms of language.

References

Anderson,R, 1988, 'Overwriting and other techniques for success with academic articles', in P.C.Robinson (ed.), *Academic Writing: Process and Product*, Modern English Publications in association with The British Council: 151-158

Borasi,R. & Rose,B.J., 1989, 'Journal writing and mathematics instruction', *Educational Studies in Mathematics 20* : 347-365

Collis,K.F., 1992, 'Curriculum and assessment: A basic cognitive model', in G.Leder (ed.), *Assessment and Learning of Mathematics*, Victoria: Australian Council for Educational Research: 24-45

Flener,F.O. & Reedy,J., 1990, 'Can teachers evaluate problem solving ability?', *Proceedings of the Fourteenth Conference of the International Group for the Psychology of Mathematics Education*, Volume 1, Oaxtepec: 127-134

Guillerault,M. & Laborde,C., 1982, 'Ambiguities in the description of a geometric figure', in F.Lowenthal, F.Vandamme, J.Cordier (eds.), *Language and Language Acquisition*, New York & London: Plenum: 151-155

Hake,R., & Williams,J., 1981, 'Style and its consequences: Do as I do, not as I say', *College English 43(5):* 433-451

Hayes,J., Schriver,K.A., Hill,C. & Hatch,J., 1992, 'Assessing the message and the messenger' *The Quarterly of the National Writing Project and the Centre for the Study of Writing and Literacy 14(2):* 15-17

Hodge,R. & Kress,G., 1993, *Language as Ideology*, 2nd edition, London: Routledge & Kegan Paul

Houston,S.K. (ed.), 1993, *Developments in Curriculum and Assessment in Mathematics*, University of Ulster, Coleraine

Kress,G., 1989, *Linguistic Processes in Sociocultural Practice*, (2nd edition), Oxford: Oxford University Press

Kress,G., 1990, 'Two kinds of power: Gunther Kress on genre', *The English Magazine 24*: 4-7

Langer,J.A. & Applebee,A.N., 1987, *How Writing Shapes Thinking: A Study of Teaching and Learning, NCTE Research Report No. 22*, Urbana, IL: National Council of Teachers of English

Lesh,R. & Lamon,S.J., 1992, *Assessment of authentic performance in school mathematics*, Washington D.C.: American Association for the Advancement of Science

MacNamara,A. & Roper,T., 1992, 'Unrecorded, unobserved and suppressed attainment: can our pupils do more than we know?', *Mathematics in School 21(5):* 12-13

Miller,L.D., 1992, 'Teacher benefits from using impromptu writing prompts in algebra classes', *Journal for Research in Mathematics Education 23(4)* : 329-340

Morgan,C., 1992, 'Written Communication: The child's perspective', short presentation at the Sixteenth Conference of the International Group for the Psychology of Mathematics Education, Durham NH

Morgan,C., 1994, 'Teachers Assessing Investigations: The Role of Algebra', *Proceedings of the Eighteenth Conference of the International Group for the Psychology of Mathematics Education*, Lisbon: 295-302

Morgan,C., 1996, *An Analysis of the Discourse of Written Reports of Investigative Work in GCSE Mathematics*, unpublished PhD thesis, Institute of Education, University of London

NCTM, 1995, *Assessment Standards for School Mathematics*, Reston VA: National Council of Teachers of Mathematics

Niss,M., 1993, *Cases of Assessment in Mathematics Education: An ICMI Study*: Dordrecht: Kluwer Academic Publishers

Torrance,H., 1995, 'Teacher involvement in new approaches to assessment', in H.Torrance (ed.), *Evaluating Authentic Assessment*, Buckingham: Open University Press: 44-56

Wade,B. & Wood,A., 1979, 'Assessing writing in science', *Language for Learning 1(3):* 131-138

LEARNING MATH IN TWO LANGUAGES

Judit Moschkovich
Institute for Research on Learning
Palo Alto, CA U.S.A.

This paper explores how Latino students construct mathematical meaning during bilingual (Spanish and English) conversations. The study addresses general questions on the nature of mathematical talk as well as questions specific to the learning of mathematics during bilingual conversations. In this paper I consider two frameworks for describing mathematics learning and its relationship to language. The first framework, a "discontinuity" model, is used to examine mathematical expressions in Spanish and English. The second framework, a "situated" model, is used to analyze two bilingual mathematical conversations between secondary students.

Although several studies have focused on discourse in monolingual mathematics classrooms (Cobb, Wood and Yackel, 1993; Pimm, 1987; Pirie, 1991), researchers have only recently begun to consider conversations in language minority classrooms (Brenner, 1994; Khisty, McLeod, and Bertilson, 1990). In general, the research on language and learning mathematics presents a view of students as facing several discontinuities: from first language to second language, from social talk to academic talk (Cummins, 1981), and from the everyday to the mathematics register[1] (Halliday, 1978).

Research specifically addressing how Spanish speakers learn mathematics in English classrooms has focused largely on students solving English word problems, rather than participating in mathematical conversations and constructing mathematical meaning. Most of this research has reflected a "discontinuity" model, describing students' problems in understanding mathematical vocabulary and in translating from English to mathematical symbols (Cocking and Mestre, 1988; Spanos, Rhodes, Dale, and Crandall, 1988). Rather than seeing learning as mapping meanings across register or language discontinuities, a "situated" model describes students' construction of knowledge as socially and materially situated-that is, viewing what students are doing as they learn mathematics as constructing meaning by using the social and material resources available to them. In this paper I consider the contributions and limitations of the "discontinuity" model and present the analysis of two bilingual conversations through the "situated" model.

"Discontinuity" model

One way to describe the role of language in learning mathematics is as the mapping of talk across discontinuities[2]. This mapping can be between two registers, between two languages, or across both registers and languages. The "discontinuity" model can be applied to describe students as learning to use new vocabulary specific to the mathematics register and to map meanings across the everyday and mathematical registers.

---

[1] A register is "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (Halliday, 1978). The mathematics register is the set of meanings, words, and structures appropriate to the practice of mathematics.

[2] While this description of the "discontinuity" model may be an oversimplification, it represents a view of the role of language in learning which sometimes appears in research studies as well as pedagogical and curricular recommendations.

EVERYDAY TALK
MATH TALK

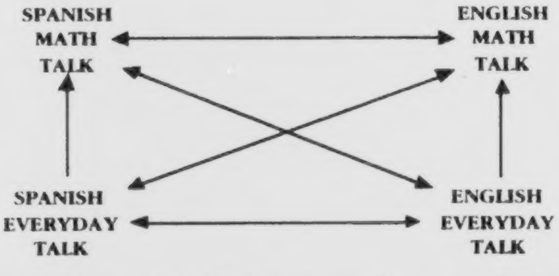
Multiple meanings for the same term can create obstacles in mathematical conversations because students often use the colloquial meanings of terms while teachers (or other students) may use the mathematical meaning of terms. Several examples of such multiple meanings have been described: "set" can mean "set the table" at home and "a set of objects" in a math context (Pimm, 1987); the phrase "any number" means "all numbers" in a math context (Pimm, 1987); "a quarter" can refer to "a coin" or to "a fourth of a whole" (Khisty, McLeod, and Bertilson, 1990); and in Spanish "un cuarto" can mean "a room" or "a fourth" (Khisty, McLeod, and Bertilson, 1990).

The discontinuity model can also be applied across two languages but within one register:

SPANISH MATH TALK
ENGLISH MATH TALK

Because there are multiple meanings for mathematical terms within the math register in each language, one mathematical term in Spanish may have several English terms associated with it. The mapping then occurs from the many associated senses and words (or semantic field) in one language to the many associated senses and words in the other language (another semantic field). For example, "menos" in the Spanish math register can be used with two different senses and in two different constructions: "minus" as in "treinta menos diez [thirty minus ten]" and "less than" as in "diez es menos que treinta [ten is less than thirty]". Students learning mathematics in these two languages would need to sort out not only the differences between two registers, but the correspondences between the math registers in the two languages as well.

The "discontinuity" model can also be applied across both registers and languages:



When students are talking math in two languages, they are not only mapping meanings across two registers or within the math register across two languages. Since the associations between words, meanings, and concepts are different in each language, they are making multiple connections across several discontinuities. For example, in English the phrase "straight line" can be associated with the everyday meaning of "straight", as used in the phrase "straight up" or "the picture is straight", meaning vertical or the opposite of crooked[3]. On the other hand, in the Spanish phrase for "straight line", "línea recta", the adjective "recta" can be associated with other mathematical objects or concepts, such as right-angledness as in "ángulo recto" [right angle]. If students wanted to describe a line as not crooked they might say "la línea está derecha [the line is straight]", which brings in other associated meanings of "derecha", such as right, meaning the opposite of left. In any case, there are multiple mappings to sort out: the difference between the two uses of the term "recta/o" within the Spanish math register, the different associations that accompany the English term "straight" in the two registers, as well as the correspondences between the meanings of "recta", "recto", and "straight."

Although the examples above focus on the uses of words and phrases, a "discontinuity" model can also include another aspect of the mathematics register, mathematical constructions. Constructions such as "if, then", "let ____ be the case", "let's assume", "this is the case because", are regularly used in explanations and arguments. There are also constructions used to describe spatial situations or make comparisons. For example, the constructions "there are four more ___ than ___" [hay cuatro más ___ que ___) and "there are four times as many ___ as ____" [hay cuatro veces más ___ que ___], refer to two different mathematical situations and yet are easily confused (especially in Spanish where "más" is used in both constructions). Students learning mathematics in these two languages would also have to map the meanings of constructions across two languages.

Although the "discontinuity" model highlights the mathematics register, it does not address other important aspects of mathematical discourse, such as the discourse practices reflected in the use of this register. In other words, the "discontinuity" model can be interpreted as reducing math talk to the use of technical vocabulary and constructions. However, mathematical discourse, in any language, involves more than the use of technical vocabulary: there are different discourse communities (mathematicians, teachers, and students) and different genres (explanations, proofs, and presentations). Within each community and genre there are practices that are part of the general characteristics of discourse. Overall, preciseness, explicitness and certainty are highly valued qualities, and abstracting and generalizing are highly valued processes in mathematical discourse. For example, claims are applicable only to a precisely and explicitly defined set of situations. Many times claims are also tied to representations. Students' participation in such discourse practices may be more evident when using a "situated" model.

The "discontinuity" model may also disregard the situational context of utterances. Although words and phrases do have multiple meanings, these words and phrases appear in talk as , utterances that occur within contexts. Much of the meaning is derived from situational resources. For example, the phrase "give me a quarter" uttered at a vending machine has a clearly different meaning than saying "give me a quarter" while looking at a pizza. The utterance "Vuelvo en un cuarto de hora" [I will return in a quarter of an hour] said as one leaves a scene has a clearly different meaning than "Limpia tu cuarto" [Clean your room], uttered while looking towards a room. When analyzing mathematical conversations, it is important to consider how resources from the situation point to one or another sense, such as whether "cuarto" means "room" or "quarter".

---

[3] Students have been observed to interpret the phrase "straight line " to mean "vertical line" (Moschkovich, 1992)

The "discontinuity" model as summarized above points to multiple meanings as possible sources for misunderstandings in math conversations and as an important aspect of learning mathematics in two languages. In this way it can frame a beginning analysis of mathematical conversations in two languages. However, this model has an unfortunate consequence: because it focuses on the conflict between registers or languages as an obstacle in learning mathematics, and fails to consider situational resources, it can easily turn into a model of students as deficient. Students' everyday experiences and first language can serve, not just as obstacles, but as resources for constructing mathematical knowledge and arriving at consensual descriptions of mathematical objects. While mathematical objects and meanings provide important resources in mathematical conversations, everyday objects or metaphors and students' first language are rich resources as well.

Situated Model

Situated perspectives of cognition (Brown, Collins, and Duguid, 1989; Greeno, 1994; Lave and Wenger, 1991) present a view of learning mathematics as participation in a community where students learn to mathematize situations and to use language to communicate about these situations. From this perspective, learning to participate in mathematical conversations (Pimm, 1987) and using social, linguistic, and material resources to construct descriptions and explanations are integral aspects of learning and doing mathematics. Within this model, language use and its relationship to math learning depend on the situation. To describe or understand a bilingual mathematical conversation we need to consider several aspects of the situation. The problem context includes whether a student is doing computation or engaged in more conceptual activities, what the sub-field of math (algebra, geometry, etc.) is, and what representational resources are available. The historical context includes a student's history with each language as well as with mathematics instruction. The social context includes who the interlocutor is and what identities or memberships are associated with each language. An example of the importance of the historical context is anecdotal evidence that people who speak more than one language carry out arithmetic computations in the language in which they learned the procedures. After completing a computation, a bilingual student may or may not, depending on who the interlocutor is, translate the answer to the other language. On the other hand, if bilingual students have not been exposed to mathematics instruction in some topics in their native language, it seems reasonable that they would talk about those topics primarily in their second language. In other situations, students might code-switch between two languages. These examples points to the importance of the situation in understanding the relationship between mathematical activity and the choice of language.

Bilingual Math Conversation Example 1: Describing a pattern

This group of students had been constructing rectangles with the same area but different perimeters and looking for a pattern relating the dimensions and the perimeter of their rectangles. In the first excerpt, the students attempt to describe the pattern in their group (Translations are in brackets and italics; for utterances in both Spanish and English, words with an English pronunciation are in italics. Teacher A speaks Spanish, Teacher B does not):

1. Alicia: Nora, dí que el más largo que está los estos ángulos, más largos que están, más el *perimeter* es . . .[*Nora, say that the longer that is the these angles, the longer that are, more the perimeter*].

2. Nora: Que? [*What?*]

3. Alicia: Más largos que están los rangulos, rángulo- [*The longer that are the* rangulos, rángulos].

4. Nora: Lo más grande que están.....[*the bigger that are*].

5. Alicia: No, largo [*no, long*].

6. Nora: Largo? [*long?]*

7. Alicia: Aha.

8. Nora: Lo mas largo que están los triángulos o como se dice....[*the longer that are the triangles or how you say it*].

9. Alicia: Los triángulos [*the triangles*].

10. Nora: *These is* triángulos [*triangles*]?

11. Alicia: Lo más [*the more*]. . . .

12. Teacher A: No, those are squares.

13. Nora: *In Spanish*, triángulos [*triangles*].

14. Teacher A: *They're* cuadros [*pictures*].

15. Nora: Cuadrados [*squares*].

16. Teacher A: Cuadrados, OK. Cuantos cuadrados [*Squares, OK. how many squares*]

17. Alicia: ¿Cómo se dice? ¿Cómo? [*How do you say it? How?*]

18. Nora: Cuadrado [*square*].

Although these students attempted to find a term to refer to the rectangles neither the teacher nor the other students could provide the word "rectángulo" in Spanish, which is the language the students were using. Later on another teacher asked several questions from the front of the class, including "How many rectangles did we find that had an area of 36?". Alicia tried to answer the last question, explaining the relationship between the length of the sides of a rectangle and its perimeter:

Teacher B: *Can somebody describe what they saw as a comparison between what the picture looked like and what the perimeter was ?*

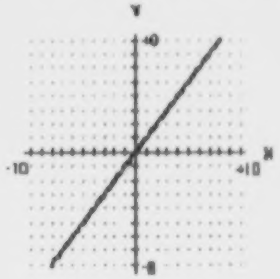
Alicia: *The longer the ah, the longer* (she gestures, tracing the shape of a long rectangle with her hands several times) *the ah, the longer the*, rángulo [*rangle*] *you know the more, the higher the perimeter is.*

An analysis of the first excerpt using the "discontinuity" model would highlight the importance of knowing a specific mathematical term, and focus on this student's failed attempt to produce the technical term "rectangle" in either language. However, if we were to focus only on Alicia's inaccurate use of the term "rángulo"[4], we might miss how her last statement is representative of mathematical discourse . Although Alicia is missing crucial technical vocabulary, she used a construction commonly used in math discourse to make comparisons and describe direct variation : "the longer the _____ the more (higher) the _______". Alicia's last utterance is thus representative of math discourse practices in a way that may not be included in an interpretation of the "discontinuity" model emphasizing technical vocabulary. Furthermore, a description of this utterance as the student's attempt to map across discontinuities between Spanish and English, would disregard her use of the situational resources available to her. Alicia interjected an invented Spanish word into her statement and used gestures to clarify her description. In this way, the "situated" model reveals how a construction, a gesture, and the student's first language can serve as resources for communicating a mathematical relationship and participating in mathematical discourse practices.

Bilingual Math Conversation Example 2: Explaining a description

These two students were working on the following problem:

If you change the equation y = x to y = -0.6x, how would the line change?



The steepness would change

Why or why not? ____NO ____YES steeper

less steep

They had graphed the line y = -0.6x on paper and were discussing whether this line was steeper or less steep than the line y = x.

22. Marcela: No, it's less steeper . . .
23. Giselda: Why?
24. Marcela: See, it's closer to the x-axis . . . (looks at Giselda) . . . Isn't it?
25. Giselda: Oh, so if it's right here . . . it's steeper right?

---

[4] Although the word does not exist in Spanish, it might be best translated as "rangle", a shortening of the word "rectángulo"

26. Marcela: Porque fijate, digamos que este es el suelo. Entonces, si se acerca más, pues es menos *steep*. [*Because look, let's say that this is the ground, then, if it gets closer, then it's less steep*].

27. Giselda: *I thought you meant* cual es la diferencia entre esto y el otro [*which is the difference between this one and that one*], *you know what I mean*?

28. Marcela: Pero fijate [*But look*].

28. Giselda : But that's not what they want.

30. Marcela: Yeah! . . . *Well, kind of, cause see this one* (referring to the line $y = x$) . . . *is* . . . está entre el medio de la *x* y de la *y* [*is between the x and the y*]. Right?

31. Giselda: (Nods in agreement.)

32. Marcela: This one (referring to the line $y=-0.6x$) is closer to the x than to the y, so this one (the line $y=-0.6x$) is less steep.

An analysis using the discontinuity model would focus on the precise meaning of the term "steep' in the mathematics register. While in math talk this meaning is related to the ratio of rise to run, this may not be the case in the everyday register (Moschkovich, 1992). Such an analysis might also focus on Marcela's use of two constructions common in the school mathematics register, "let's say this is..." and "if ____, then ____" in line 26. However, Marcela's explanations in lines 26, 30, and 32 (in bold) are representative of math discourse in several ways that go beyond the use of such constructions and are more evident from a "situated' perspective. First, Marcela explicitly states an assumption, a characteristic practice in mathematical discourse, when she says: "Porque fijate, digamos que este es el suelo." [*Because look, let's say that this is the ground].*" Second, she makes a connection to the representation, another practice characteristic of mathematical discourse, using the line $y = x$ (line 30) and the axes (line 32) as reference objects to support a claim about the steepness of the line. And lastly, she uses the metaphor that the x-axis is the ground: "Digamos que este es el suelo. *[Because look, let's say that this is the ground,]* as a resource in explaining her description. Marcela thus used resources from the everyday context, a metaphor comparing the x-axis to the ground, and from the mathematical context, the line $y = x$ and the axes. The "situated" model has thus been useful in describing how this student, rather than struggling with the discontinuity between the everyday and the mathematical contexts, used resources from both contexts to construct a mathematical explanation.

Conclusions

The "discontinuity" model serves to clarify multiple meanings in math conversations and provides some analytical power for describing learning mathematics in two languages. However, this model has several limitations. It can be interpreted as reducing math discourse to the use of technical vocabulary and it fails to consider situational resources. Because it focuses on the conflicts and obstacles between registers or languages, it can become a "deficiency' model. The situated model can serve to broaden the analytical lens to include more aspects of the situation. This perspective also generates different questions, such as what resources students use. The two examples presented above show that, while mathematical objects and meanings do provide resources in mathematical conversations, everyday objects or metaphors and students' first language can be resources as well. In this way the "situated' perspective can show how the everyday context and students' first language might be resources, rather than obstacles, for learning mathematics.

References

Brenner, M. (1994). A communication framework for mathematics: Exemplary instruction for culturally and linguistically diverse students. In B. McLeod (Ed.), *Language and Learning: Educating linguistically diverse students*. Albany: SUNY Press.

Brown, J.S., Collins, A., Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18 (1), 32-42.

Cobb, P., Wood, T., and Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In N. Minick, E. Forman, and C. Stone (Eds.), *Education and Mind: Institutional, social and developmental processes.* New York: Oxford University Press.

Cocking, R. and Mestre, J. (Eds.), (1988). *Linguistic and cultural influences on learning mathematics*. New Jersey: LEA.

Cummins, J. (1981). The role of primary language development in promoting educational success for language minority students. In California State Department of Education (Ed.), Schooling and language minority students: A theoretical framework (pp. 3-49). Los Angeles: Evaluation, Dissemination and Assessment Center, California State University, Los Angeles.

Greeno, J. (1994, August). The situativity of learning: Prospects for syntheses in theory, practice, and research. Paper presented at the American Psychological Association, Los Angeles, CA.

Halliday, M. A. K. (1978). Sociolinguistics aspects of mathematical education. In M. Halliday, *The social interpretation of language and meaning*. London: University Park Press.

Khisty, L.L., McLeod, D., and Bertilson, K. (1990). Speaking mathematically in bilingual classrooms: An exploratory study of teacher discourse. Proceedings of the Fourteenth International Conference for the Psychology of Mathematics Educator, Vol. 3, 105-112. Mexico City: CONACYT.

Lave, J. and Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, England: Cambridge University press.

Moschkovich, J. N. (1992). Making sense of linear graphs and equations: An analysis of students' conceptions and language use. Unpublished doctoral dissertation, University of California, Berkeley.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London: Routledge.

Pirie, S. (1991). Peer discussion in the context of mathematical problem solving. In Durkin and Shire (Eds), *Language in mathematical education: Research and practice*. Philadelphia: Open University Press.

Spanos, G., Rhodes, N., Dale, T.C., and Crandall, J. (1988). Linguistic features of mathematical problem solving: Insights and applications. In R. Cocking and J.P. Mestre (Eds.), *Linguistic and cultural influences on learning mathematics*. New Jersey: LEA.

HERMENEUTIC EXPERIENCES IN CONSTRUCTING LESSONS AND CLASSROOM RESEARCH

Judith A. Mousley

Deakin University

Eight teachers were given the same instructions for a mathematics lesson. Their interpretations, and those of their pupils, demonstrated moments of decision making which resulted in eight very different lessons, demonstrating distinct patterns of teaching and learning. This paper outlines how hermeneutic philosophy relates to teachers' and students' interpretations and re-interpretations of this text. It claims that the hermeneutic situation of the researcher is also open to exploration.

Introduction

The phenomenological orientation that this project draws on is hermeneutics, which is based on the assumption that we cannot divorce ourselves from our own reason and historical contexts. It argues the hermeneutic position that community-based experience and the possibilities that people (in this case teachers and researchers) see for alternative actions are inseparable factors—and connections between these are open to exploration.

In 1986, Wachterhauser claimed that hermeneutics is "a family of concerns and critical perspectives that is just beginning to emerge as a program of thought and research" (p. 5). Nevertheless, hermeneutics originated in ancient Greece, where it was recognised that people bring different exegeses (*hermenëia*) to an event, and that studying these assisted knowledge development. Recognition of this link between interpretation and learning prevailed through the Middle Ages, where it was used in the study of biblical texts.

Schleiermacher (1838/1977) initiated a German branch of this phenomenological orientation by reaching beyond the interpretation of text to examining common consciousness of kind. He introduced the notions of community and history to this field. At the beginning of this century, Dilthey (1972) took up these points, proposing that "expression"—a product of an individual's historicity—characterises interpretation, and that all social phenomena are the expressions of persons whose rationality is structured by community-based individual realities.

> To summarise Dilthey's theory: the task of the interpreter—and also the purpose of hermeneutics—is to unite the past with the present through a process of reconstruction. In this way, the connections between expressions, experience, structure of meaning, and life are clarified. (Ödman, 1988, p.66)

More recently, Heidegger (1962) further developed the notion of historicity. He claimed that we understand only through a contingent situatedness of a continually-changing world that is not of our own making—the world of our historically-mediated culture—and that our understandings constitute our very being. Understanding is based on pre-understandings which evolve as we re-develop our social contexts. A key element of his thesis was the hermeneutic circle:

> The hermeneutic circle involves the 'contextualist' claim that the 'parts' of some larger reality can be understood only in terms of the 'whole' of that reality, and the 'whole' of that reality can be understood only in terms of its parts. That is to say that understanding any phenomenon means, first of all, situating it in a larger context in which it has its function and, in turn, it also means letting our grasp of this particular phenomenon influence our grasp of the whole context. (Wachterhauser, 1986, summarising Heidegger's claims, pp. 23-24)

Another element in hermeneutic philosophy is the importance of language. According to Heidegger (1962), it is the source of pre-understandings, and "conceals within itself a developed mode of conceiving" (p. 199). It lives and grows as a response to reality, but also serves to shape that reality.

This notion was furthered by Gadamer (1960), who claimed that our framework of language-based prejudices (pre-judgements) "constitute the initial directedness of our whole ability to experience" (p. 245). They develop through living in communities, mediate perception, and form an historical basis for rational activity. Westphal (1986) pointed out that the key to unlocking such traditions is the language of an historic community, claiming "because language is always some specific language and never language in general, we find reflection on consciousness turning into reflection on language, which in turn become reflection on tradition" (p. 70).

The research project reported here explored decision-making moments in the teaching of an activity. It focused on the way that one short instruction was understood variously by different teachers and their students. I wish to argue that interpretations of the original and developing texts at particular stages of the lesson were framed and shaped by the subjects' contextual histories—and my interpretations of the resulting data were a product of my own social history.

Method

Five female teachers and three male teachers of classes from Years 1-6 from three State schools in a lower middle-class area were involved in this study. Each was met separately by the researcher, and asked to teach a given activity. It was written on an index card:

From a given piece of cardboard, make a regular shape which holds one cup of birdseed. Make a similar shape which is twice as big.

Data collection from each class involved field notes made straight after the teacher was given the task and after pre-teaching interviews a few days later; videotaping of the teacher throughout the lesson; and videotaping, audio taping or observation of groups of students during the seatwork section of the lesson. Most importantly, the teachers were shown snippets of their videos two weeks after the lessons and asked to talk about their intentions with regard to particular teaching actions.

Analysis was aimed at describing how classroom participants' roles had been construed and reconstrued throughout the activity—the teachers' initial interpretation, planning, presentation, control and assessment of the task as well as the students' re-interpretation and management of the activity. The researcher aimed to describe the sense that classroom participants made of their social world—the meanings that people give to their environment. These meanings were examined in the light of Berlak and Berlak's (1981) sixteen "dilemmas of teaching". Thus two levels of interpretation were involved—the subjects' renditions as well as the subjective readings of these by the researcher.

Some results

Construing of the task by teachers and planning of lessons both involved discourse with self. Similarly, control of classroom activity was discursive, creating moments where students needed to construct understandings. Their discussions, too, needed to be interpreted by other students. At these moments, language-based understandings were shaped by the past, present and perceived future social contexts of each of the participants. I use the word "moments" to reflect the immediate nature of split-second periods in which the potential of any lesson is re-shaped. Identified moments were:

Teachers	Students
Interpreting	
Planning	
Presenting -------------------------->	Interpreting
Supervising	Planning
Evaluating	Performing
	Presenting
	Evaluating

This paper focuses on the arrowed link between teachers' presentations of the task and students' re-interpretations of it. It was found that the former impacted markedly on options pupils had for the latter. Let me demonstrate this with three upper-primary-school examples.

Teacher H, Year 6: In the pre-lesson interview, this teacher expressed concern that the given task may not lead to the discovery of the "mathematical knowledge bound within the task". His instructions to pupils, written and read aloud, were "Make a box which holds one cup of seed. Make another box which holds twice as much." A week after the lesson, the researcher (R) interviewed this teacher (TH):

431 R I am interested … to know why you thought the shape should be a box. Had you thought about the possibility of making other shapes?

432 TH Yes, but I wanted to build on this lesson to give them an understanding of volume.

433 R Good. So they will do that with box shapes?

434 TH Yes. They have to learn length by width by height and … well, they couldn't do that with other shapes. … Oh, I guess they could, but, like cones and other shapes - I didn't want shapes where they couldn't measure length and width and height.

In planning the activity around a particular learning objective—a formal rule—Teacher H expected all students to take a directed path of "discovery". His pupils demonstrated an acceptance of this context, displaying characteristics of students waiting to be led. For example:

463 Rob But it has to be a box. Not a box. He said a cube. That's the same all around. The same size - this way, this way, this way. Ask him how big. Darren, ask him how big to make it.

464 Darren How big would fit. You've got the cardboard. How big could we make it? It has to hold a cup.

465 Rob Just ask him, Daz. He knows.

466 Darren Okay. He knows. (*Inaudible*) Mr H…

Teacher N, Year 5: This teacher wanted pupils to discover what happens when all dimensions of a cube are doubled. Her instructions were oral. She first told the children to "Make a cube 5 cm by 5 cm by 5 cm". When they had all finished, she asked them to make one measuring 10 cm by 10 cm by 10 cm and to use the birdseed to compare the volumes of their shapes.

Her students achieved her aim but also displayed dependence:

134 Rhana It's too big. We were only meant to double it. You went wrong, Karen

135 Karen No. They are right. 5 cm this one, 10 cm that one. You try it … (*inaudible*) … With the ruler.

136 Rhana Mmm, but there's too much seed. It should be two lots. Why is there seven and a half? It's mad. Two lots would be right. Don't write it down. I'm going to ask her if we did it right.

Following Rhana's question, the teacher called the class to attention. After a short discussion which led to the teacher expressing the generalisation that "The big one holds eight times as much", she gave a six-minute explanation of why this is so. Gestures were used frequently and terms such as *third dimension*, *multiply*, *multiples* and *comparative volume* were used. When asked later if she thought that all of the children would have understood, she claimed:

231. TN Yes. That is why it is important to have the hands-on work first. Yes. They had seen it with their own eyes. That is why children need to do real things in mathematics - so they understand why the mathematics works.

Teacher M, Year 5/6: On seeing the activity, this teacher said:

730 TM It will be interesting to see what they make of it. I'm not even going to explain what *regular* means. I wonder if they will ignore the word. ... Seeing the different shapes will be fun. And their reactions when they double different aspects.

Teacher M presented the task as given, writing the instructions on the board, then reading them aloud. The children could work in groups or alone.

One group of three girls built a square pyramid, left unsealed at the apex so seed could be poured in using a funnel. They found that their shape held more than one cup of seed, so traced it onto another piece of card and trimmed the triangular parts of the net gradually until a pyramid of the right size was formed.

748 Silvia Don't take too much off. Remember ... (*inaudible*) ... you are not just taking that bit. That long bit. You are taking it four times. There ... and there ... and (etc.)

749 Binny And it's not just thinner. The shape. Look. It gets shorter so you are losing this bit. The top bit every time. (*Some trial and error followed*)

783 Silvia Good. That's it. One cup. I'll tape it up. ... Now. Two cups. Now for two cups.

784 Rachael Two cups. Yes. Or twice as big? Two cups isn't twice as big.

785 Binny (*Inaudible, then laughed.*)

786 Silvia Yes it is. But I know. But ... I know what you think. Like ... like two times the edges. Make the bottom twice as big. The sides too?

787 Binny Not the sides.

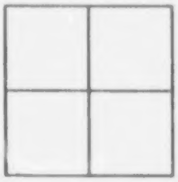
788 Silvia Why?

789 Binny We make the bottom twice as big. Right? Longer and wider. But that will be like this ... and this ... and this ... and this. Like four times as big. We need to decide if we want it really twice as big ... or each side double - fourth as big. Anyway, then we could cut down the sides like we did ... so it holds two cups.

790 Silvia Mmm. I get it. That way it's twice as big. Well, on the edges anyway.

791 Binny What edges? What are you on about?

792 Silvia These edges. The sides. The sides of the square. See. Look. You've got a square, right? (*drew a square.*) Then you double this one ... and that one; and you don't have two. What do you have?



793 Binny Four. I know that.

794 Silvia Good. Four. (*inaudible*) So do we want twice ... or four times? Hey? I reckon only twice. Twice as big.

795 Binny Yeah, and ...

796 Silvia And it holds two cups. So its twice as big like that.

797 Rachael But it needs to be twice as high.

798 Silvia Look, it doesn't say twice as high on the board. Or two cups. Look.

799 Rachael No.

800 Silvia We need to decide ... to make up our minds. What are we going to call twice as big. Jees, I wish we'd started with a box.

801 Binny No, a box is no good. They are doing a box. Anyway, it's the same. We would be the same. Look, if you do this to the box ... (Binny drew a sketch of a box then one twice as long) ... you've got ...

802 Rachael Two cups. But it's not twice as big.

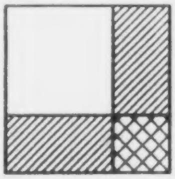
803 Silvia Yes. Yes it is.

804 Rachael But it's not twice as wide ... or high ... just long. Forget the box. I think we should not double the sides. Of the bottom. We should work out what would make twice as big really. Like. The square. The area ... (*inaudible*) ... of the square.

805 Silvia Yeah. Mmm.

806 Binny Times it ... Times it by one and a half. Two is too big because you get four. No. One and a half ...

807 Silvia That's too big. One point two five - or less? No, probably more ... that ... one and a quarter. It's not one and a half. It's less. Look. If you've got this square (*drew a square*). Right? And it's half as long again ... and half as long on to this side (*hatched areas*). Right? That's one and a half? But then you've got this piece (*double-hatched areas*). That means one and a half is too much. It's more than two.



808 Rachael Can we work backwards? Jees. No. (*long pause*) We don't know how many little squares. Get some graph paper so we can do it on little squares. Like , we count the squares. Then double it.

809 Silvia Yeah. Then work out how long the sides need to be.

810 Binny Would that make it hold two cups?

811 Silvia No. Rachael, where's the graph paper?

Discussion

For Gadamer, the universal and the particular are co-determined and hence can be understood only in relation to each other. The socio-historical context of the first two lessons are familiar to us all who share traditions of teacher-centering, objectivism, and hierarchical forms of control. Teachers' and children's expressions of such histories (the parts) and many others like them contribute to the whole contexts, i.e. to interaction in that classroom, to mathematics education and more broadly to the nature of schooling. But at the same time the parts are shaped by that whole. Our community histories form powerful constraints to thinking and acting otherwise.

Similarly, the historical context of the third classroom is familiar. This teacher has developed what many would call "constructivist" ways of working; and is very articulate about her constructivist style and how she developed it progressively over the last ten years. Once, the "whole" of this context would have been almost unimaginable, to her or to us, but now we see such contexts as possible and desirable. In other place and other times they may not be seen in this light.

Well after the completion of my data analysis, it is useful to reflect on its own hermeneutic aspects. Why did I choose to focus on these incidents (and similar ones for each of the moments listed above), then interpret them as being typical of particular pedagogical styles? Why did I explore them in relation to Berlak and Berlak's control dilemmas and what were the effects of this? What are the silences in my work and why do they exist? What particular incidents from my past, present, and planned future histories set the horizons of the reality from which I view the world?

I remember early discussions with Glen Lean about Platonist versus Constructivist strands of the philosophy of ideas, and can recall other vital dialogues and readings as I struggled to come to grips with this area. Similarly, current philosophies of mathematics education expressed in texts and other professional conversations, and dominant ideologies expressed in dialogues within my Faculty, as well as papers and tutorials where I tested emerging ideas all played their parts in this development. These and other experiences provided new possibilities—and hence potential for the interplay between reality and possibility. On the other hand, my reality constrained possibility by its very existence.

Space does not allow more detailed dialogue about the context of the inquirer. It is clear, though, that my research actions (and possibilities as well as lack of possibilities for alternative approaches) were shaped by my beliefs—in turn carved by interest in the hegemony of classroom control—itself a product of my schooling and identifiable aspects of my professional community life. The context of my work is a product of time and places. While it is a property of individuality, that individuality has been developed in a community with its own particular history.

Conclusion

Gadamer claimed that authentic understanding is not detached from the interpreter but constitutive of his or her *praxis*. He stressed that understanding can only be an act of interpretation through the life-world of the interpreter. Text, for instance, written or spoken by one partner in the hermeneutic conversation, is expressed only through its interpreter (Gadamer, 1975). Such claims have implications for us as teachers and as researchers.

It was not difficult to find many examples of hermeneutic situations in the classroom interactions. These became useful starting points for dialogue with the teachers about their interpretations of their own actions. They were able to link their actions with particular beliefs and habits, embedding these in their professional training or experience as well as in current institutional and social contexts. Hermeneutics provides a useful framework for discussing such moments, but it was not surprising that teachers' and students' understanding, interpretation and application were clearly linked with each other and with their social contexts.

However, what has really intrigued me in my reading on hermeneutics is the possibility of applying the notion of historically-determined contexts to my own work as a researcher. Links between language, experience, and the possibilities I see for alternatives have become exciting new areas for exploration. Just as the instructions given to (and by) teachers were texts, my data and written accounts are texts not to be taken as givens. These are objects—which Gadamer calls "universal" in that they are now open to many particular readings—to be understood.

> ... the interpreter seeks no more than to understand this universal thing, the text... In order to understand that, he must not seek to disregard himself and his particular hermeneutical situation. He must relate the text to his situation, if he wants to understand at all. (Gadamer, 1972, p. 289)

References

Berlak, A., & Berlak, H. (1981). *Dilemmas of schooling: Teaching and social change.* London: Methuen.

Dilthey, W. (1972). *New literary history 3: The rise of hermeneutics.* F. Jamieson (Trans.) Cambridge: Cambridge University Press.

Gadamer, H-G. (1960/1989) *Truth and method.* J. Weinsheimer & D.G. Marshall (Trans). New York: Crossroad Press.

Gadamer, H-G. (1972). *Kleine shriften III.* Tübigen: J.C.B. Mohr.

Gadamer, H-G. (1975). *Truth and method.* New York: Seabury.

Heidegger, M. (1962). *Being and time.* J. Macquarrie & C. Robinson (Trans.). New York: Harper & Row.

Ödman, P.J. (1988). Hermeneutics. In J. Keeves (Ed.), *Education research methodology and measurement: An international handbook* (pp. 63-69). New York: Pergamon Press.

Schleiermacher, F.D.E. (1938/1977). *Hermeneutics: The handwritten manuscripts.* H. Kimmerle (Ed.), J. Dule & J. Forstman (Trans.). Missoula, MA: Scholars Press.

Wachterhauser, B.R. (1986). History and language in understanding. In B.R. Wachterhauser (Ed.), *Hermeneutics and modern philosophy* (pp. 5-64). Albany, NY: State University of New York Press.

Westphal, M. (1986). Hegel and Gadamer. In B.R.Wachterhauser (Ed.), *Hermeneutics and modern philosophy* (pp. 65-86). Albany, NY: State University of New York Press.

YOUNG STUDENTS' INFORMAL KNOWLEDGE OF FRACTIONS

Hanlie Murray, Alwyn Olivier and Piet Human
Research Unit for Mathematics Education
University of Stellenbosch, South Africa

Research shows that young children come to school with a rich store of informal mathematical knowledge. It has been found that for whole number arithmetic this knowledge should be built on and that children's informal methods lead to correct and powerful computational methods. It is felt that children's informal knowledge of fractions should be used in the same way. This study explores young children's informal knowledge of fractions and suggests equal sharing situations as appropriate links to young students' intuitions.

Introduction

Several teaching experiments have shown that it is possible to elicit and build on young children's conceptualizations of computational problems and the strategies they construct based on these conceptualizations. Instead of ignoring or even actively suppressing children's informal knowledge, and imposing formal arithmetic on children, such instruction recognizes, encourages and builds on the base of children's informal knowledge. This is made possible and even easy because young children's informal knowledge about whole numbers (and their understanding of problem situations involving whole numbers) is strong and almost completely free of misconceptions (e.g. Murray & Olivier, 1989; Murray, Olivier & Human, 1991, 1994).

It has been suggested that young children also possess similar informal knowledge about fractions, which should be used in the same way (e.g. Steffe & Olive, 1991; Baroody & Hume, 1991).

However, there is evidence of misconceptions about fractions among elementary school students. D'Ambrosio and Mewborn (1994), for example, document a number of misconceptions (which they call *limiting constructions*) and possible reasons for these limiting constructions in young students. Some of these limiting constructions clearly originated in the child's pre-school or outside school experiences. For other limiting constructions it is difficult to determine the role that teaching might have played, especially inthe case of older students (e.g. Baroody & Hume, 1991; Pirie & Kieren, 1994; Steffe & Olive, 1991).

The limiting constructions that are not caused by teaching may have different sources. For example, at a certain level of development the young child does not discriminate between units and parts of units, and will count three whole chocolate bars and half a bar as "four chocolates" (a developmental phase problem), whereas children who accept half of an object as *any* big fractional part are simply responding to the inexact but functionally sufficient treatment of such parts of objects in the home (a problem with words which have very exact mathematical meanings but which are used loosely and vaguely in everyday situations).

This study

This study forms part of our on-going research project on problem-centered learning and teaching of mathematics (e.g. Murray, Olivier & Human, 1991, 1994).

The main purpose of this study was to gain more information on the *kind* of informal knowledge of fractions and problem situations involving fractions that young children bring to school. Knowledge about fractions involves knowledge about the *concept*, of which two subconstructs are the part-whole relationship between the fractional part and the unit, and the idea that the fractional part is that quantity which can be iterated a certain number of times to produce the unit (D'Ambrosio & Mewborn, 1994). It also involves knowledge of the fraction *names* and of the fraction *symbols*. Since it is possible that lack of knowledge of the fraction names and symbols may prevent young students from demonstrating their intuitive concept of fractions, we decided to create *problem situations* to which students could respond, rather than asking questions or presenting test items which use fraction names and/or fraction symbols (e.g. Neuman, 1993). A number of equal sharing problems with remainders which also had to be shared out, were formulated. No fraction materials or manipulatives (e.g. clay) were made available to students; only unlined paper and crayons.

A first grade and a third grade classroom from a small school were involved in the study. There were 22 students in each class. The parent population is lower to upper middle class. Although these students' whole number arithmetic teaching is based on a problem-centered approach where students are confronted with a large and planned variety of problem situations that they solve through individual effort, discussion and reflection, the study of fractions in the first two grades was up to then limited to teacher-led discussion of halves and quarters based on activities involving paper folding and pre-partitioned manipulatives.

The researcher (Murray) posed two problems to the students during the fifth month of the school year. The students solved the problems individually. Individual students then explained their thinking to the researcher or the teacher. These annotated solution strategies were used as the basis for the following analysis.

First graders' solution strategies

The following problem was posed to the six weakest students (as perceived by the teacher):

> *Three friends have to share four vienna sausages among themselves so that nothing is left and they each get an equal share. How must they do this?*

The other students solved the same problem with seven sausages. We will refer to these problems as 4 ÷ 3 and 7 ÷ 3.

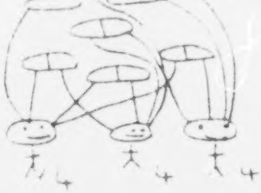
The teacher had not discussed fractions at all up to then.

None of the students referred to the fractional parts as anything other than "pieces" or "bits," but where appropriate distinguished between "bigger pieces" and "smaller pieces." All students used direct representations (in this case drawings of the friends and the sausages) to solve the problem.

We summarize below the students' solution strategies, *tentatively* presented in ascending order of sophistication. We illustrate each strategy with an example.

1. Partitioning some units (i.e. whole sausages) so that the total number of units and fractional parts is a multiple of the divisor. (Frequency: 1)

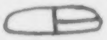
   Brigitte solves 7 ÷ 3 by keeping two units whole, cutting five units in half and then sharing out the 12 "objects" equally among the three friends, ignoring the size of the object. She says: "Each friend gets four sausages."

2. Sharing out the maximum number of units, then partitioning the remaining units into inappropriate fractional parts (i.e. not multiples of the divisor), so that when the fractional parts are shared, one friend gets more parts. (Frequency: 1)

   For 4 ÷ 3, Eustace gives each friend one unit, partitions the remaining unit into quarters, and shares out one quarter to each of two friends and two quarters to the third friend. He explains: "Each boy gets one sausage and one piece, but this boy (pointing) gets one sausage and two pieces." He says that the sharing is unequal, but that this is his best plan.

3. Sharing out the maximum number of units, then partitioning the remaining units into the correct *number* of fractional parts which are however of unequal size. (Frequency: 3)

   For 7 ÷ 3, Charlene gives each friend two units, then partitions the remaining unit into three pieces as follows  , and gives each friend a piece.

4. Sharing out a mixture of units and different fractional parts. (Frequency: 3)

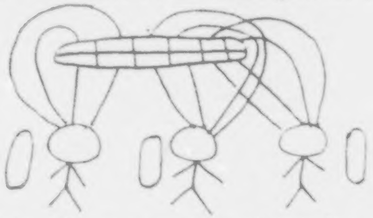
   For 7 ÷ 3, Anel gives each friend one unit. She then partitions two units in half and gives each friend two halves. She partitions the remaining unit in thirds and gives each friend a third. She explains: "Each friend gets a sausage and two bigger pieces and one smaller piece."

5. Sharing out some units and an appropriate number of equal-sized fractional parts. (Frequency: 1)

   For 7 ÷ 3, Pierre gives each friend a unit, then partitions the remaining four units into thirds. Each friend therefore receives "one sausage and four small pieces."

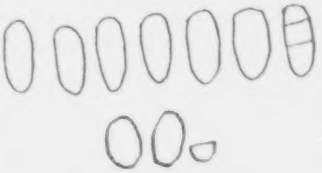
6. Sharing out the maximum number of units, then partitioning the remaining units into a number of equal parts which is a multiple of the divisor. (Frequency: 1)

   For 4 ÷ 3, Chester gives each friend one unit, partitions the remaining unit into 12 pieces, and says each friend gets "one sausage and four crumbs."

7. Sharing out the maximum number of units, then partitioning the remainder into the minimum number of suitable fractional parts. (Frequency: 11)

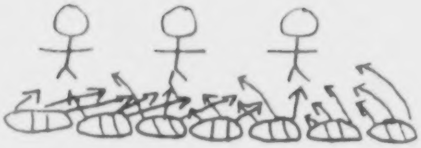
   For 7 ÷ 3, Johan explains: "Two sausages and a piece. I cut this sausage into three pieces. If you put the pieces together, they make *one*, *one* sausage."

8. Partitioning all the units into fractional parts of which the denominator is the same as the divisor, then sharing out the parts from each unit in turn. (Frequency: 1)

For 7 ÷ 3, Jenni partitions each sausage in thirds and says that each friend receives seven pieces, one piece from each sausage.

She also states that you can put six of these pieces together to make two whole sausages.



Third graders' solution strategies

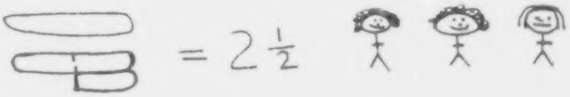
The 7 ÷ 3 problem was posed to the whole class.

These students had yet not studied fractions in grade 3, but had studied them as described in the previous grades.

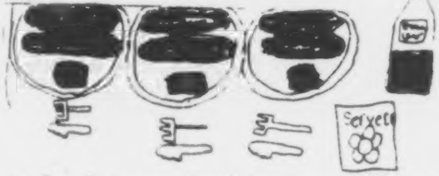
Students were requested to also make a drawing of their answer even if they believed that they could solve the problem numerically without drawing.

We again summarize the strategies.

1. The solution is drawn correctly (two units and a third of a unit), but the units and fractional part are counted together, giving "3 sausages" as answer. (Frequency: 1)
2. Sharing out the maximum number of units, ignoring the remaining unit. (Frequency: 1)
3. A solution of $2\frac{1}{2}$ units is clearly drawn and named "$2\frac{1}{2}$." (Frequency: 3) Koba:

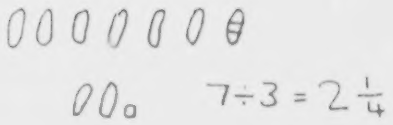


4. The correct solution is clearly drawn, but not named. (Frequency: 3) Jay draws this picture:



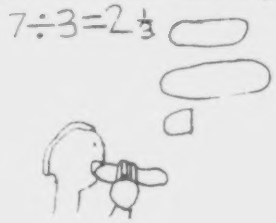
5. The correct solution is clearly drawn, but incorrectly named as $2\frac{1}{2}$ or $2\frac{1}{4}$. (Frequency: 11)

Marco:



6. The correct solution is clearly drawn and named. (Frequency: 3)

Chris:



Discussion

The Grade 1 solution strategies have some interesting features. Firstly, these students seemed to have had even less exposure to the fraction *concept* and especially to the fraction *names* than we expected. No student tried to name the fractional parts; they simply referred to the parts as "pieces," sometimes as "bigger" and "smaller" pieces. Previous experience and discussions with teachers had led us to expect that "half" was commonly misused as a name for any big fractional part, and "quarter" for a small part. No student in this group did so.

We also believed that the idea of "equal sharing" was problematic for young children, partly because sharing situations within a family usually involve family members of different ages and that "fair" sharing in these cases actually imply *unequal* shares, and partly because even attempts at equal sharing in the home are frequently only approximate (cf. D'Ambrosio & Mewborn, 1994). Most of these Grade 1 students fully understand equal sharing, with 17 out of the 22 students constructing equally shared solutions. Also, none of the students ignored the remainder or tried to get rid of it ("Give the extra sausage to Mother").

Lastly, we found it surprisng that only one student (Brigitte) did not distinguish between units and fractional parts, and concentrated on the number of "pieces" given to each friend instead of on the "amount" of sausage.

One difference between the first and the third graders' responses is that four third grade students produced *wrong* answers, as opposed to incorrectly labelled answers (solution strategies 2 and 3). This seems like an outcome of their instruction where the exclusive use of halves and quarters had led to a limiting construction: They provided answers of $2\frac{1}{2}$ and 2 because they could not conceptualize any other possibility, whereas the first graders' solutions utilized a variety of fractional parts.

Another difference is the high incidence of incorrect *naming* of the fractional part among the third graders, although their drawings clearly show that the students are dealing with thirds and not halves or quarters (a frequency of 11). One reason for this may be that instruction failed to make students understand that halves and quarters are not *general* labels for fractional parts, but signify *particular* fractional parts. The fact that they had met only halves and quarters in their teaching probably strengthened this misconception.

This study shows that first graders *do* have "a wealth of informal knowledge on which we can base the teaching of fractions" (Steffe & Olive, 1991). These young children clearly have informal knowledge about the fraction *concept*, enabling them to *understand and solve* sharing problems involving fractions. Knowledge of the fraction names and symbols (even halves) seems to be very little, but the incidence of limiting constructions also seems to be very low. Only one first grader failed to discriminate between units and fractional parts, and another four produced unequal (but only slightly unequal) shares.

We hypothesise that the limiting constructions which have been identified may largely be the results of teaching and the use of particular materials. The third grade responses point towards the dangers of limiting students' experience to only some fractions (halves and quarters) and also to the limited success of paper folding and similar activities as environments in which to construct the concept of fractions (cf. Pothier and Sawada, 1990).

On the other hand, the first graders' responses show that equal sharing situations elicit ideas about partitioning units into equal parts and about combining parts to form units – both ideas are crucial subconstructs of the fraction concept (the part-whole and the iterative-part-to-form-a-whole subconstructs).

We therefore suggest that many of the older students' limiting constructions may be prevented by introducing fractions in the lower elementary school through posing sharing problems with remainders that also have to be shared out. This encourages young students to construct their own idea of fractions through their own actions. Different solutions can then be compared and discussed, the teacher gradually introducing the necessary terminology and notation.

This programme is already followed in the lower elementary grades of some local schools (cf. also Empson, 1995 and Streefland, 1993). The use of sharing problems should, however, only be regarded as an introduction; students also need problem situations which embody the other meanings of fractions (fraction as ratio, as operator, etc.).

References

Baroody, A. J. & Hume, J. (1991). Meaningful Mathematics Instruction: The Case of Fractions. *Remedial and Special Education, 12* (3), 54–68.

D'Ambrosio, B. S. & Mewborn, D. S. (1994). Children's Constructions of Fractions and their Implications for Classroom Instruction. *Journal of Research in Childhood Education, 8* (2), 150–161.

Empson, S. (1995). Using Sharing Situations to Help Children Learn Fractions. *Teaching Children Mathematics*, October, 110–114.

Murray, H. & Olivier, A. (1989). A Model of Understanding Two-Digit Numeration and Computation. In G. Vergnaud, J. Rogalski & M. Artigue (Eds.), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education, Vol. 3* (pp. 3–10). Paris, France.

Murray, H., Olivier, A., & Human, P. (1991). Young children's division strategies. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education, Vol. 3* (pp. 49–56). Assisi, Italy.

Murray, H., Olivier, A. & Human, P. (1994). Fifth Graders' Multi-digit Multiplication and Division Strategies After Five Years' Problem-Centered Learning. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, Vol. III*, (pp. 399–406). Lisbon, Portugal.

Neuman, D. (1993). Early Conceptions of Fractions. A Phenomenographic Approach. In I. Hirabayashi, N. Nohda, K. Shigematsu & F. Lin (Eds.), *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education, Vol III* (pp. III-170–III-177). Tsukuba, Japan.

Pirie, S. & Kieren, T. E. (1994). Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments. *For the Learning of Mathematics, 14* (1), 39–43.

Pothier, Y. & Sawada, D. (1990). Partitioning: An Approach to Fractions. *Arithmetic Teacher, 38* (4), 12–17.

Steffe, L. P. & Olive, J. (1991). The Problem of Fractions in the Elementary School. *Arithmetic Teacher, 38* (9), 22–24.

Streefland, L. (1993). Fractions: A Realistic Approach. In T. P. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational Numbers: An Integration of Research*, 289–326. Hillsdale, N. J.: Lawrence Erlbaum Associates.

Tensions in the Novice Mathematician's Induction to Mathematical Abstraction

Elena Nardi

Oxford University Department of Educational Studies

The ongoing doctorate project, on which this paper is based, is a study of the novice mathematician's conceptual and reasoning difficulties in their encounter with mathematical abstraction. For this purpose 20 Oxford first year mathematics undergraduates have been observed and audio-recorded in their weekly tutorials and interviewed twice in a period of two academic terms. The Oxford syllabus topics, which the tutorial and interview content draws on, are Linear Algebra, Continuity-and-Differentiability, Topology, Sequences-and-Series and Groups-Rings-and-Fields. Data analysis is now in process and aims at the emergence of data grounded theory. Here I present two potentially problematic aspects of the novices' induction to mathematical abstraction. Both observations constitute parts of the emerging theme on the nature of the novice mathematician's enculturation into the thinking and acting within the realm of Advanced Mathematics.

The theoretical origins of the study, on which this paper draws, lie in the realisation that an educational reform regarding mathematics teaching cannot take place in the absence of an awareness of the learner's thought processes. Coupled with the intrinsically idiosyncratic epistemological complexity of mathematics, the cognitive dimension of didactics arises as particularly significant [1].

With regard to learning advanced mathematics this study originates in the assumption, grounded on the relevant literature (e.g. [8]), that a novice mathematician faces a series of cognitive difficulties in the encounter with mathematical abstraction. As noted in previous presentations of parts of the study (e.g. [6], [7]) abstraction is meant both from a psychological perspective i.e. that the advanced mathematics learner has to build up knowledge in an axiomatic way and learn how to reason deductively; and from an epistemological perspective, i.e. that the nature of the objects of advanced mathematical learning can extend beyond the physical or the numerical.

In the above, learning is not seen as isolated in a cognitive vacuum but embedded in a sociocultural context [10]. Therefore, in a constructivist strand of thinking [5], the learner's cognition, while being personal and individually interesting, is also emphatically seen as taking place in a learning environment. In this case the context within which learning takes place is the Oxford undergraduate mathematics course. This study seeks to construct a psychological profile of the novices' difficulties in their encounter with mathematical abstraction by probing into their expressions of learning. It is assumed here that cognition can only become visible and accessible through the learners' oral and written (in this study: oral) articulations of their mathematical thinking. As a thought process cognition is esoteric and inaccessible. In fact this is a phenomenological study of advanced mathematical cognition [3].

Experience from a Pilot Study to this study [6] provided evidence that the tutorials given to first year mathematics students in Oxford can be a substantial source of data regarding the novices' expresssions of mathematical cognition. Observation, audio-recording of the tutorials and interviewing of the observed students were chosen as the qualitative techniques through which access to these expressions would be achieved. Observation of tutorials on *Linear Algebra, Continuity-and-Differentiability, Topology, Sequences-and-Series* and *Groups-Rings-and-Fields* was relatively unsystematic but informed by the cognitive aims of the study as well as by research in the field of advanced mathematical thinking. It lasted 14 weeks and approximately 200 hours. Interviews were carried out twice and they were minimally structured around mathematical topics that during observation emerged as particularly problematic for the learners. The openness of the selected methodological techniques was a natural consequence of the decision to ground the theory generated by this study on the data [4].

Data analysis takes place at the time this paper is written. Tutorial and interview recordings have been transcribed and tabulated in terms of their mathematical content and their didactical content. The tutorial material has been repeatedly scanned and, via a gradually more selective process, a number of crucial learning episodes has been extracted as the pivotal material for the analysis. The rest, called non-episodic material, is used as supportive material that enriches and contextualises the episodes. The latest selection resulted in the extraction of approximately 70 crucial learning episodes.

The interview material consists of collections of open individual discussions with the students on six particularly problematic topics: *accumulation/isolation points/openness/closedness, limit, spanning sets, compactness, convergence of series and sequences* and *the First Isomorphism Theorem for Groups/relevant concepts.*

Further analysis is currently arranged in the form of five sections on the Foundations *of Analysis, Calculus, Topology, Linear Algebra* and *Abstract Algebra* . Each of the 70 learning episodes mentioned above is presented and analysed as a text. Text here is the conglomeration of the recording, the transcript, the notes taken during observation and the contextual documents (problem sheets, lecture notes, reading lists). Analysis of the text is supported by the non-episodic material and the interviews. The psychological observations on each episode are collected and presented as a totality in the end of each section. The final synthesis of the cross-topical theoretical abstractions of the study is based on the intermediate theorising that takes place in the five topical sections.

The Novice's Induction to Mathematical Abstraction: an Uneasy Encounter

As an example of the cross-topical themes that constitute the final theorising of the study - or as a flavour of the outcomes that are now in the process of emerging - I refer here to some aspects of the novices' transition to advanced mathematical thinking as observed in the tutorials. This transition signifies the quintessential cognitive shift that learners are expected to undertake in the first few months of university studies. Therefore in a sense, and given their secondary school mathematical background, these students' transition from school to university mathematics stands as a metaphor for the transition from concreteness to abstraction and from empirical/inductive explanation to deductive proof and axiomatics. Endorsing the Piagetian discourse on Reflective Abstraction and the Vygotskian approach to learning in a sociocultural context, this transition is seen here as a social and psychological enculturation process [2]. This paper highlights some of the tensions generated in this cultural exchange between the Expert (the tutor) and the Novice (the student).

In the four Extracts cited below the focus is on the tentative induction of the novice to the style and essence of formal mathematical thinking. Inevitably the tutors' perceptions of this style and essence influence dramatically their responses to the students' style and thinking. Here I draw on two kinds of tension observed in the exchange:

T1. The tension between verbal/explanatory and formal mathematical expression.

T2. The tension between Proof via Basic Principles and Proof via Quoting Proved Statements.

I present the Extracts and briefly discuss.

Tension I: The tension between verbal/explanatory and formal mathematical expression

Extract 1.1

Middle of Second Term. Individual Tutorial to Student Jack. Series and Sequences.

> *They are looking at Jack's draft. His answers have been generally correct but the tutor comments several times on his writing not being clear and precise. In one of the questions the tutor and Jack agree on using Cauchy's Criterion[1] and the tutor asks Jack what is a Cauchy sequence. Jack replies:*
>
> ***J: ...for every n bigger than N... any small...It's the difference between any two terms after a certain limit is less than epsilon.*** [J1]
>
> *Later on the tutor complains about Jack's writing on another question[2] from the same problem sheet:*
>
> *T: One thing I couldn't understand is where did you get these numbers from. Er,... you didn't seem to explain your answer properly to... I wasn't sure how to explain where these numbers came from...*
>
> ***J: Yeah,. that was brief...*** [J2]
>
> *T: I think you know what you are doing because you got the right answer which is terminal... but...can you explain to me...*
>
> *Then Jack describes what sequence n_r looks like: he explains how he first removed all the numbers containing the digit 7 from the tenths, then the hundredths and so on; first removing numbers ending in 7, then the others. At the same time he was careful with not removing the same number twice. He then noted that $\sum 1/n_r$ was convergent by comparison. The tutor listens and in the end he approves:*

---

[1] a_r is a Cauchy sequence if $\forall \varepsilon > 0\ \exists N \in \aleph$ such that $\forall n,m > N$ $|a_n - a_m| < \varepsilon$.

61

> *T: Right... but as I said none of this was written and you know it could have been a little ... but OK if you feel alright about this, it is right and solemn after what you said,... but ... Do you want me to go through the whole thing on the board or you feel you knew what was happening in there?*
>
> ***J: Well, my problem is with the mathematical writing of this... I mean you can's write something like this in the Mods It's not the idea, it's the mathematical writing[3]...*** [J3]

The incident starts with the tutor's comment on Jack's problematic writing. However it is J1, Jack's attempt at explaining what a Cauchy sequence is, that illustrates graphically a tension in Jack's mind: Jack attempts a reconstruction of the formal definition (...for every n bigger than N... any small...I), fails and resorts to what he seems to be quite familiar with: ordinary words employed to present his conception of what a Cauchy sequence is.

His verbal explanation is a semi-formal statement that conveys a concept image [9] of a Cauchy sequence as a sequence whose terms are coming infinitesimally close to each other. In J1 verbal explanation wins over formal definition. In other words in Jack's concept image personal interpretation of the definition is a stronger presence and a much more easily retrievable piece of information than the formal definition.

The tutor initiates the discussion on mathematical writing and Jack agrees (J2). The tutor then encourages Jack to expand on his explanation in a precise manner. He approves of the thinking behind Jack's writing but is very firm about conveying to Jack the standard requirement for adequate explanation in a mathematician's writing instead of dry sequences of numbers on paper . Given that Jack has not appeared to be reluctant or incapable of justifying his intuitions, it seems that what keeps him away from putting his justifications on paper is a concern about the acceptability of ordinary discourse in the presentation of a mathematical argument.

Accustomed to wordless, notation-laden telegraphic proofs from lectures and textbooks, Jack seems to confuse logical formalism with linguistic formalism ignoring that the latter is merely a semantic aid to the former. Jack's concern (J3) has, apart from the metamathematical one outlined above, a practical dimension too: he is worried that the use of ordinary language in a demonstration of a mathematical argument is not acceptable in exams. However his concern and hesitation, despite stopping him from elaborating in writing on the question talked about in Extract 1.1, are signs of a formal mathematical consciousness in genesis. Jack seems to begin being preoccupied with formal mathematical expression: J2 and J3 are indications that in Extract 1.1 Jack does not merely pursue an answer to a mathematical question - actually he already has one - but to questions of mathematical form and convention.

---

[2] Let n_1, n_2, n_3 be the subsequence of the natural numbers with any number containing the digit 7 omitted. Show that $\Sigma(n_r)^{-1}$ converges.

[3] Mods is an abbreviation for Moderations, the examination taken by undergraduates in the end of their first year.

In the following Extract 1.2, Carly expresses a similar concern. Remarkably it is with the student's initiative this time, not the tutor's, that the issue of mathematical writing is brought up; that maybe the reason why Carly's dissatisfaction is more persistent.

Extract 1.2

End of First Term, Pair Tutorial to Students Carly and Charlie (silent) Continuity and Differentiability.

> *Carly is dissatisfied with her answer on a question*[4]*:*
>
> ***C: I didn't really answer it properly... My answer was not mathematical, it was more like an essay... I could see it was true but I couldn't see how to prove it mathematically.***
>
> *She then describes what she did:*
>
> *C: If you differentiate this, you get that the* x^{n-1} *dominate and ....f is monotonic. So if n is even, again* x^{n} *dominate and if n is even then it is above the axis always and there is no root...But...*
>
> *She sighs. The tutor then responds:*
>
> *T: But you see a lot of maths is not about writing down equations but about stating logically connected arguments... And in fact the fewer symbols you've got written down, the more words... and especially if you've got to do a [names lecturer and examiner and laughs]... then that's much better. So what you said was probably right.*
>
> *Carly silently looks down at her notes and a few seconds later she asks:*
>
> ***C: Well, how do you do it?***

Despite the tutor's initial, yet unjustified, approval of Carly's approach and the generally encouraging tone of her response, Carly is not convinced. The tutor's philosophical and affective treatment may be seen as not satisfying Carly's cognitive demand for a response

to the particular question and
to her general metamathematical concern with the 'essay' type of argumentation.

I note here that both Extracts come from tutorials well into the 14-week observation period (11th and 7th respectively). In the meantime the students have been repeatedly advised against relying on intuitive arguments and non formal reasoning. So both Jack and Carly seem to be beyond the stage of relying on empiricism and are now struggling with their definitions of formalism. In this struggle the semantic formalism dominating their mathematical experiences of the first months in the course seems to yield in them the impression that prose is not an efficient tool for presenting, in the tutor's words, *logically connected arguments* .The tutor's role here seems to be the resolving of this misunderstanding. In other words the tutor appears as the cultural mediator of *acceptable manners* in the context of formal mathematical communication.

Tension II: The tension between Proof via Basic Principles and Proof via Quoting Proved Statements.

Turning to, and at the same time developing suspicion and a slight aversion towards, verbal explanations was described in the previous section as the novice's state of mind when standing between school empiricism and university formalism. In this section I point at another tension that goes beyond the semantic and into the heart of the novices' decision making regarding their reasoning practices: the tension between employing Proof Via Basic Principles and Proof Via Quoting Previously Proved Statements. In Extracts 2.1 and 2.2 (and the corresponding figures[5]) I cite a tutor's and a student's approach to two questions appearing in the course's weekly problem sheets. The juxtaposition of the approaches aims at highlighting the differences between an Expert's and a Novice's approaches to mathematical reasoning.

---

[4] Prove that the equation $1+x+x^2/2+...+x^n/n=0$ has no real root if the positive integer n is even and precisely one root if n is odd.

Extract 2.1

End of Second Term, Pair Tutorial to Students Carly and Charlie, Series and Sequences.

> *The tutorial begins with a discussion of the students' responses to a question[6] on the convergence of some series. The tutor notes that Charlie, for i and ii, and Carly, for iii, have correctly evaluated the infinite sums by 'splitting them up"* [7] *in two known infinite sums but he also recommends the more 'formally acceptable' way of doing the same first on the finite sums and then taking the limit. He then asks Carly to present part iv (see fig. 1a). He agrees and suggests an **alternative approach** (see fig.1b) which, he concludes, is **a widely used technique.***

(a) Carly's Way	(b) The Tutor's Way
${}_1\Sigma^{\infty} r^2/3^r = {}_1\Sigma^{\infty} r^2\text{-}1+1/3^r = {}_1\Sigma^{\infty}(r-1)(r+1)/3^r$ $S_\infty=1/2$ $={}_1\Sigma^{\infty}(r+1)(r-1)/3^r+1/2$ $= {}_0\Sigma^{\infty}(r+2)r/3^{r+1}+1/2 = 1/3\ {}_1\Sigma^{\infty} r(r+2)/3^r + 1/2$ $= 1/3\ {}_1\Sigma^{\infty} r^2/3^r + 1/3\ {}_1\Sigma^{\infty} 2r/3^r + 1/2$ $2/3\ {}_1\Sigma^{\infty} r^2/3^r = {}_1\Sigma^{\infty} 2r/3^{r+1} + 1/2$ $= {}_1\Sigma^{\infty} 3r/3^{r+1} - r/3^{r+1} +1/2$ $= {}_1\Sigma^{\infty} r/3^r - r/3^{r+1} + 1/2$ $= (1/3-1/9+2/9-2/27+3/27\ldots) +1/2$ $= (1/3+1/9+1/27+1/81\ldots) = 1/2 +1/2$ so $\Sigma r^2/3^r = 3/2$	Note that if $f(x) = {}_1\Sigma^{\infty} x^r = 1/1-x$, then $f'(x) = {}_1\Sigma^{\infty} rx^{r-1} = 1/(1-x)^2$ and $f''(x) = {}_1\Sigma^{\infty} r(r-1)x^{r-2}$ Then by writing f" in terms of f' and f, and for $x=1/3$, it turns out that $\Sigma r^2/3^r = 3/2$.

Figure 1

From a topical point of view (in this study *topical* means exploring the cognition of a particular mathematical concept) Carly's solution probably generates an interest with regard to her handling of the series as if they are finite sums. Luckily she escapes paradoxical situations because the sums she deals with are finite limits. Hence the liberties she is taking with rearranging the terms turn out to be harmless.

Here however the focus is on the students' reasoning. In Carly's refreshing back to basics approach, ostensibly, the only piece of previous knowledge she employs is that $\Sigma 1/3^r=1/2$. This is however a deceptive appearance since behind Carly's rearrangements lies the theory that makes them possible. However her approach, though not terribly elegant, is pragmatic and straightforward. It has the feel of handy arithmetic and shows skill and imaginative capacity. On the other hand the tutor's approach is a formal and elegant shortcut in resonance with the material the students have been taught at lectures and the techniques they will need. It has the benefit of hindsight and of globality. It shows an expert handling, an informed awareness of the facilities available to the craftsman ($\sum x^r=1/1-x$, letting $f(x)$ be $1/1-x$, calculating f' and f'' and noting that f'' can be written in terms of f and f') as opposed to Carly's decontextualised, hence slightly primitive approach (writing r^2 as r^2-1+1, moving 1/3 inside and outside the $\sum$ several times, etc.).

---

[5] where I reproduce the solutions presented on the blackboard by students and tutors as duplicated in notes made during observation.

[6] Evaluate the following infinite sums, giving reasons for your answers: (i) $\Sigma 1/r(r+k)$, where k is an integer, $k\geq 1$, (ii) $\Sigma 1/r(2r+1)$, (iii) ${}_{-\infty}\Sigma^{+\infty} e^{-|x+r|}$ $(0\leq x\leq 1)$, (iv) $\Sigma r^2/3^r$.

[7] e.g. by noting that $1/r(r+k)=1/kr-1/k(r+k)$.

None of the above is meant to diminish Chrisitna's efficient approach which (the dangers of naive rearrangement of the terms in a series aside) yields the correct answer. It only aims at highlighting the inclination of the novice to resort to familiar (here: handling of algebraic expressions) modes of operating. Similar signs of prematurity are given in Extract 2.2.

Extract 2.2:

End of Second Term. Pair Tutorial to Students Carly and Charlie. Groups-Rings-and-Fields.

> *In the same tutorial as in Extract T2.1, a bit later, Carly is invited by the tutor to present one of her answers on the blackboard*[8]*. Slightly reluctant she accepts and warns:*
>
> ***C: Oh, OK but you may not like it!***
>
> *'Does it matter?' replies the tutor. In fig2a I present her solution and in fig.2b the alternative solution suggested by the tutor.*

(a) Carly's Way Abbreviated	(b) The Tutor's Way
Let $X=o(x)$, $Y=o(y)$ and $t=o(xy)$. By commutativity she shows that $(xy)^{XY}=e$. Then: $XY=mt$ for an integer m. Hence: $XY\geq t$. By commutativity and $hcf(X,Y)=1$ she shows that $t=nXY$ for an integer n. Hence $t\geq XY$. Therefore: $t=XY$.	He recalls that, in case of commutativity and $hcf(o(x),o(y))=1$, $o(xy)=o(x)o(y)/\|\langle x\rangle\cap\langle y\rangle\|$ implies $o(xy)=o(x)o(y)$ because then $\langle x\rangle\cap\langle y\rangle=\{e\}$.

Figure 2

Significantly Carly is a bit reluctant to present her solution; given that the incident in Extract 2.2 follows the incident in Extract 2.1 it seems reasonable to consider the possibility that Carly begins to suspect that, though correct, her approach is not exactly up to the standards of elegance and resonance with the material she has been taught recently. She doesn't say *it's wrong* or *you may reject it*. She says: *you may not like it*. . Signs of a developing taste for a certain mode of reasoning appear.

Similarly to Extract 2.1 Carly resorts to a solid arithmetical handling (to prove that integers a and b are equal, it is sufficient to prove that $a\leq b$ and $b\leq a$). The tutor on the other hand employs a theorem and invests the arithmetical relationship given in the question ($hcf(o(x),o(y))=1$) with its group-theoretical meaning ($\langle x\rangle\cap\langle y\rangle=\{e\}$).

Again the juxtaposition highlights the differences between an expert and a novice approach. There are some redundancies in Carly's way as well as some unclarified points - not necessarily visible in the abbreviated version in Fig.2a. Most important though maybe her starting to conceptualise the need (not merely aesthetic but mostly instrumental) for an embedded, contextualised mode of reasoning and for an organically connected argumentation, in a way which will turn the coherence and connectedness of mathematical theories to her benefit. In other words via this juxtaposition of approaches, she might begin learning about the benefits of mathematical expertise.

---

[8] Check that if $xy=yx$ in a group, and $hcf(o(x),o(y))=1$, then $o(xy)=o(x)o(y)$.

Towards a Theory of the Novice's Learning

In the brief discussion of the incidents cited above a flavour was given of the type of cognitive observations made in the study. In these examples of intimate learning interaction the tutors seem to hold a key-role as mediators of *appropriate cultural behaviour* : they illustrate the power of language in demonstrating a mathematical argument and they provide refined alternatives to the students' mathematical arguments. These arguments, verbal/intuitive (Extracts 1.1 and 1.2) and basic/slightly decontextualised (Extracts 2.1 and 2.2), highlight some aspects of the learners' state of thinking in the early stages of their studies. Their metamathematical concern about the validity of these arguments (particularly in Extracts 1.1 and 1.2) can be seen as signs of an emerging formal mathematical consciousness. A didactical preoccupation originating in the above is then how mathematics teaching at this level can reinforce the growth of this consciousness in an informed and systematic way .

References[9]

1. BALACHEFF N (1990). Future perspectives for research in the psychology of mathematics education. In KILPATRICK J, NESHER P, INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education. Chapter 7: 135-148. Cambridge University Press, UK.

2. BISHOP A J (1991) Mathematical Enculturation. Kluwer Academic Publishers.

3. CURTIS B, MAYS W (1978) (eds) Phenomenology and Education. Methuen, London.

4. GLASER B G & STRAUSS A L (1967). The Discovery of Grounded Theory: strategies for Qualitative Research. New York, Aldine de Gruyter.

5. GLASERSFELD VON E (1987) Learning As A Constructive Activity in JANVIER, C (ed.) Problems of Representation in the Teaching And Learning Of Mathematics , USA: Lawrence Erlbaum Associates, Inc., Publishers.

6. NARDI E (1994) *Pathological Cases of Mathematical Understanding* Proceedings of the 18th Annual Conference of Psychology in Mathematics Education. Lisbon, Portugal.

7. NARDI E (1995)*The Novice Mathematician's Encounter With Mathematical Abstraction: The Case of Accumulation Point.* Proceedings of the European Research Conference on the Psychology of Mathematics Education. Osnabrück, Germany.

8. TALL D (1991) *The Psychology of Advanced Mathematical Thinking*. Chapter 1 in TALL D (ed) Advanced Mathematical Thinking. Kluwer Academic Publishers.

9. TALL D and VINNER S (1981). "Concept image and concept definition in mathematics with particular reference to limits and continuity." Educational Studies in Mathematics 12: 151-169.

10. VYGOTSKY L S (1962) Thought and Language. Cambridge MIT Press.

---

[9] I would like to warmly thank my supervisor Dr Barbara Jaworski for her comments on the first draft of this paper.

STUDENTS ASSESSMENT OF AN ALTERNATIVE APPROACH TO GEOMETRY

L. Nasser (IM/UFRJ); N. Sant'Anna (Colégio Pedro II); A. P. Sant'Anna (UFF)
Projeto Fundão — UFRJ — Brazil

An alternative approach to secondary school geometry in Brazil has been developed, and tested in some schools in Rio de Janeiro, for six years now. Although the positive results of the former trails in terms of students performance and teachers approval, we had no idea about the students opinion. In this paper we describe our proposal, based on the 'Van Hiele Model of Thinking in Geometry', and the inquiry conducted in order to get the students assessment of the course. The results show a positive approval from the students, both in a global point of view, and in each topic of geometry focused in the 7th- and 8th-grades.

Introduction

The difficulties shown by secondary students in a systematic Geometry course is a well-known problem, pointed out by teachers and researchers from several countries along the last decades. Clements and Battista (1992), for instance, claim that:

> *As we have seen, and belying its obvious importance in the curriculum, students' performance in geometry is woefully lacking. Neither what students learn in geometry nor the methods by which they learn it are satisfactory.* *(p. 457)*

Among the suggestions that appeared to alliviate these difficulties, it seems that the most investigated and commented one is ***The van Hiele Model of Thinking in Geometry*** (van Hiele, 1986). The van Hiele model suggests that students progress through a developmental sequence of conceptual understanding as they learn geometry. Although various aspects of the theory have been investigated, there are still some controversial issues, such as the discreteness or continuity of the levels (Gutierrez, Jaime and Fortuny, 1991); the designing of reliable tests to identify students' van Hiele levels (Usiskin and Senk, 1990); the place of the hierarchical inclusion of classes in the van Hiele theory (De Villiers, 1987) and the dependence on teaching strategies (Malan, 1986). An overview of the van Hiele model, including the summary of the criticisms and research carried out, is provided by Clements and Battista (1992).

The van Hiele theory is the framework adopted by our research group in the attempt to develop an alternative approach to the traditional Geometry course in which 7th- and 8th-grade Brazilian students (13-15 year-olds) are engaged. The activities, specially designed, have now been tested for six years, with a positive approval from the teachers. According to the data collected from the first sample, along the subsequent years of schooling, the students exposed to the experimental treatment overperformed the ones on the control group (Nasser and Sant'anna, 1995).

Nevertheless, we did not have any hint about the students opinion about the approach adopted. Teachers' approval does not indicate for sure that the students also enjoy the focus given to the course. An inquire to investigate the students point of view about this dynamic and constructivist way to teach geometry was necessary.

In this paper we describe how the van Hiele theory was used to guide the development of this alternative approach to secondary school geometry. We also analyse how students evaluate this approach, according to their answers to a specially designed questionnaire.

The teaching of Geometry in Brazil

Many researchers have commented about the decline on the teaching of geometry, all over the world. In the UNESCO book dedicated to the 'Teaching of Geometry' (Morris, 1986), almost all articles by Mathematics Educators from several countries point out this decay, as Lange:

> *... interest in geometry has declined throughout the decades. And, in secondary school education especially, the lack of interest has so eroded the subject that it has even disappeared completely from the curriculum of some schools.* *(1986, p. 59)*

Tall (1995) also worried about this, arguing that "*the decline of Euclidean geometry in English schools has led to a loss of experience with systematic proof*" (p. 73), which, in his opinion, may be an obstacle for students' long term development of creative learning.

In Brazil, the same decline has been noticed in the last three decades. In particular in the first grades, very few Geometry is taught, and not systematically. As primary teachers do not have a special training in Mathematics, they, in general, do not like and do not know enough Geometry to teach it, avoiding this subject.

Geometry is taught systematically at secondary school, in the second half of grades seven and eight. The approach is very traditional: Euclidean Geometry, with emphasis on proof, and very few concrete materials are used. Since there is not a national curriculum in Brazil, there are slight differences from one region to another, concerning secondary school Mathematics. The differences are rather in the sequence in which the topics are presented than on the topics themselves. In general, there is no reference to three-dimensional geometry in the programme. Some special 3-D shapes appear in the 5th grade programme, as a means to introduce the measurement of volume of the cube, cuboid and cylinder. The representation of points by coordinates in the plane appears in the 6th grade programme. But transformations are not taught at all at secondary school. They appear later in the programme, in some special schools. Also vectors and matrices are taught only at this higher degree.

Because teachers have to teach at several schools and various grades, they do not have enough time to prepare their lessons, and the common practice is to adopt a book to be followed. So, theory and exercises can be found in the textbook and very often the number of math's lessons is not enough to cover all the book. As geometry comes at the end of the textbooks, some topics on geometry can be missed. Some experimental approaches to Geometry have been suggested by education projects. For example, the Fundao Project has produced a booklet suggesting the introduction of Geometry through 3-D solids, which has been used in some schools in Rio de Janeiro since 1985. According to the teachers, children are more likely to be motivated to study Geometry with this approach, than with the traditional one, introduced by point, line and plane.

The alternative course

The geometry course proposed is developed in the framework of the van Hiele and the constructivist theories. This means that all the activities are designed following the descriptors of the van Hiele model, respecting the levels attained by the students, and are to be applied in a constructivist perspective, in which students work in groups, taking an active role in the learning process.

In our proposal, Geometry is introduced according to the approach mentioned above, through the 3-D space. At the beginning of the 7th-grade (13-14 year-olds), the students are tested in order to have their van Hiele levels identified. In general, in a class, we find students reasoning at the first and second levels, but, in some schools where geometry has not been taught previously, it is possible to have a great number of students without level, i.e., who have not reached even the first level. Some special activities have been designed in order to help students to upgrade their levels, bringing the class to become more homogeneous (Nasser, 1992).

The topic of *congruence of plane shapes* is one of the most problematic for 7th-grade Brazilian students, who are not prepared to reason in a deductive way. In fact, while the majority of them are reasoning at the first two van Hiele levels, the exercises in the textbook ask for proofs, using the cases of congruence of triangles (SAS, SSS, ASA), which requires more advanced reasoning. So, the topic of congruence illustrates the mismatch between the van Hiele level in which the instruction is given and the levels really attained by the students, which is one of the causes of the difficulties. The study of congruence requires reasoning at least at the third level, since the objects of study are the relations between shapes (Nasser, 1990). Therefore, we propose activities in which congruence is taught at the third van Hiele level, diminishing the gap between the van Hiele levels attained by the students, and the one adopted for the teaching.

In order that the instruction of congruence be kept at this level, deduction and proof must be eliminated. Transformations are used instead, to justify the congruence of shapes, through the conservation of shape and size. As Transformations are not included in the Brazilian secondary school syllabus, activities about Reflections, Translations and Rotations have been designed. These activities are developed at the third van Hiele level, according to the level descriptors for Transformation Geometry suggested in Nasser (1989). In the didactical experiment to test this material carried out with the first sample, in 1990, in two schools in Rio de Janeiro, the experimental students presented better results than the control ones, who followed the textbook (Nasser, 1992).

For the 8th-grade, the **enlargement tranformation** is used to introduce the concept of similarity, following the same approach. Students deal with the comparison of shapes and objects, such as Coke bottles, TV and cinema screens of several sizes and photographs, looking for the necessary and suficient conditions for similarity. This proposal for the 8th-grade has now been tested for three years in about five schools, and after some adjustments received the approval of the teachers involved on the trials.

Students assessment

Up to the moment, in order to evaluate the geometry course proposed, we only worried about the students performance and the teachers approval. Now, we decided it was time to investigate the students opinion about the course. Students' agreement is crucial for the success of a didactical experiment (Santos and Nasser, 1995). Informally, we had the idea that the pupils were enjoying the geometry course, regarding their behavior in class and their expontaneous presence in extra lessons.

In general, teachers do not pay much attention to students' assessment, arguing that they tend to evaluate teachers who give good marks positively and underevaluate the rigorous teachers, who really make them learn. Nevertheless, the use of objective questions has proved to be a valuable tool to avoid these deviations. Calling students' attention to the objective aspects of the quality of the course, it is possible to obtain a reliable global evaluation, as desired.

In order to get the assessment of the course from the students point of view, a questionnaire has been designed, in terms of comparison with the teaching of other disciplines. The student is asked to compare, under several criteria, the approach adopted in the geometry course with the teaching and learning of other topics of Mathematics and other disciplines. This comparison is made by means of a five steps' scale, in which students are asked if each topic of the geometry course was Much Worse (-2), Worse (-1), Similar (0), Better (+1) or Much Better (+2), when compared to the traditional approach. In order to avoid bias from the students, the questionnaire was applied at the end of the school year, after the teacher has decided the final marks, but before these were announced to the students.

The questionnaire was composed of two parts. In the first one, the student was asked to give a global evaluation of each topic of the course. For the 7-th grade sample, the topics focused were: geometric solids, transformations, quadrilaterals and congruence of shapes. For the 8th-grade students, the following topics were included:

scales, enlargement, similarity and right angled triangles. The second part of the questionnaire involved four qualities related to the teaching of each topic of the geometry course. The student was asked to give a mark from the five steps' scale for each topic, regarding if the way the topic was taught fosters the development of each of the qualities:

I- comprehensive learning
II- upgrading of the van Hiele level;
III- active participation at classroom activities;
IV- students' motivation through the use of suitable resources.

Being aware that the questionnaire was not simple for the students to answer, the teacher read it with the class, explaining each item. The 7th-grade sample had 70 students, while 88 8th-grade students answered the questionnaire, at the end of the 1994 schoolyear.

Analysis of the results of the assessment

The global evaluations given are shown in tables 1 and 2. The high percentage of positive evaluations (better and much better) of the methodology under judgment give a clear view of the students approval.

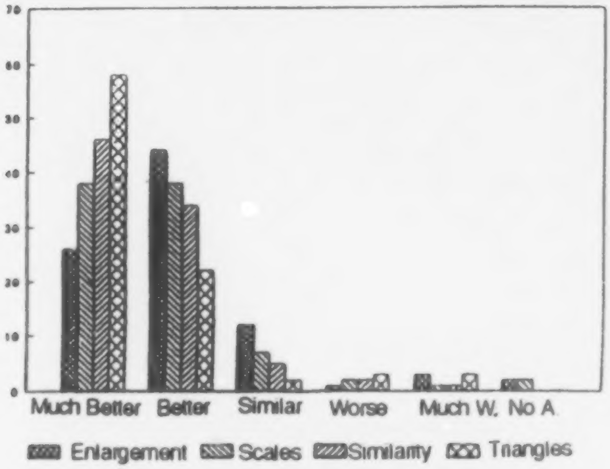
Table 1: 7th-grade answers (N=70)

Topic	Much worse	Worse	Similar	Better	Much better
Solids	1	0	1	17	50
Transformations	0	2	1	25	40
Quadrilaterals	0	2	0	17	50
Congruence	1	1	6	25	35

Table 2: 8th-grade answers (N=88)

Topic	Much worse	Worse	Similar	Better	Much better
Scales	1	2	6	36	37
Enlargement	3	1	12	44	26
Similarity	1	2	5	34	45
Right angled triangle	3	3	2	22	58

It is worth noticing the indications that the approval increases as the students are exposed to more geometrical concepts through the alternative methodology. This indication comes from the analysis of the marks given in each topic of the 8th grade program, as shown in graph 1 below. The large number of "Much Better" evaluation is reached at the last topic of the program: right angled triangles.

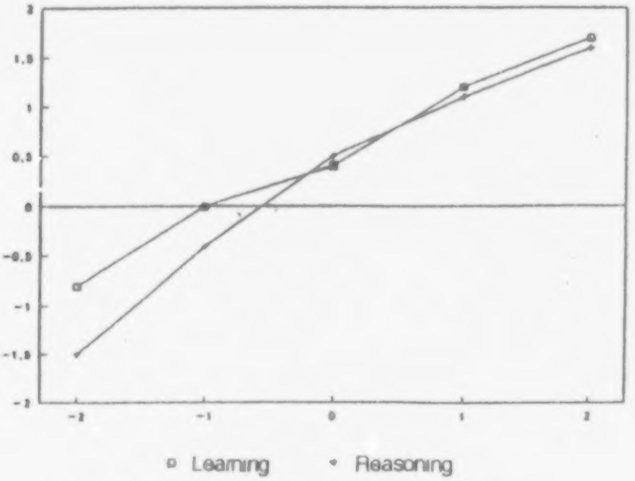


Graph 1: Final evaluation by subject

The objective items of the questionnaire were designed taking into account a model for the student's global evaluation, which links this evaluation to his assessment of the results of the teaching-learning process, by means of the acquisition of the concepts and of the upgrading of his van Hiele level. These results are, otherwise, explained in two ways: through the use of suitable concrete resources and through the increase in the participation of the student in class.

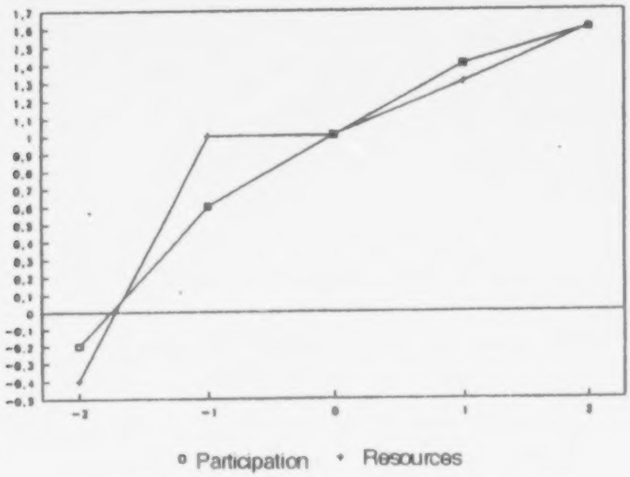
The answers to the questionnaire fit this model, showing that the students completely approve the methodology adopted in the geometry course, according to the analysis developed by means of the statistics software STATGRAPHICS.

Graph2 shows the straight relation linking the final appraisal to the evaluation of the conceptual learning and upgrading of the reasoning level for the 8th-grade students. Similar results are obtained for the 7th-grade.

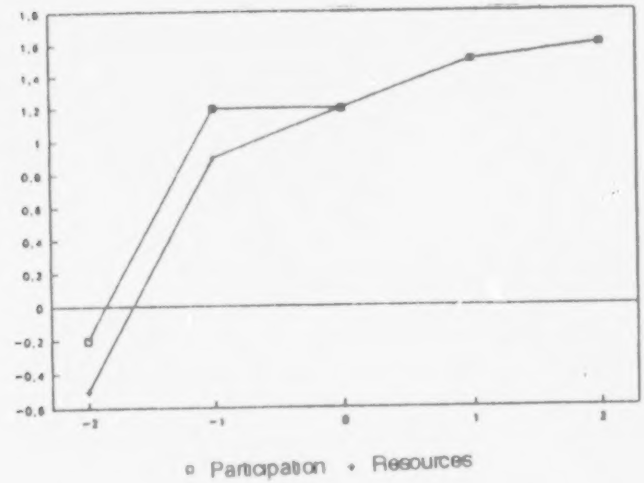


Graph 2: Explanation of final evaluation

The relationship between the evaluations given in terms of conceptual learning and its explanatory variables: use of resources and participation in class are illustrated by graph 3. Analogously, graph 4 shows the relationship between the evaluations in terms of upgrading the reasoning level and the same explanatory factors. Both graphs are built on 8th-grade data; the data of the **7th grade** follow the same pattern.



Graph 3: Explanation of conceptual learning



Graph 4: Explanation of reasoning level

Conclusions

The analysis of the answers shows that the great majority of students evaluate the approach given to all the concepts as better or much better than the traditional one.

There is high coherence between the global evaluation and the assessment by means of concept learning and the upgrading of the van Hiele level, which indicates the students' satisfaction with the course.

High coherence is also found between the marks given to concept learning and reasoning level, by means of the active participation in class and the use of didactical resources. The presence of these components in the methodology used can be associated to the better results of learning, and corroborates the coherence of the students' answers.

References

Clements, D. H. & Battista, M.T. (1992): Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning,* pp. 420-464. New York, NY: Macmillan Publishing Company.

De Villiers, M. (1987): *Research evidence on hierarchical thinking, teaching strategies and the van Hiele theory: some critical comments.* Paper presented at Learning and teachimg geometry: issues for research and practice working conference. Syracuse, NY. Syracuse University.

Gutierrez, A., Jaime, A. & Fortuny, J.M. (1991): An alternative paradigm to evaluate the acquisition of the van Hiele levels. JRME, 22, pp. 237-251.

Lange, J.J. (1986): Geometry in the primary school: what is possible and desirable. In: R. Morris (Ed.) (1986), *Studies in Mathematics Education, Vol.5, Teaching of Geometry.* UNESCO.

Malan, F.R.P. (1986): *Teaching Strategies for the Transition of Partition Thinking to Hierarchical Thinking in the Classification of Quadrilaterals: Some Case Studies.* Internal Report No.3, University of Stellenbosch, RUMEUS, South Africa.

Morris, R. (Ed.) (1986): *Studies in Mathematics Education, Vol.5, Teaching of Geometry.* UNESCO.

Nasser, L. (1989): Are the van Hiele Levels Applicable to Transformation Geometry? *Proceedings of PME-13,* Vol 3, pp.25-32, Paris, France.

Nasser, L. (1990): Children's Understanding of Congruence According to the van Hiele Model of Thinking. *Proceedings of PME-14,* Vol. 2, pp. 297-303. Oaxtepec, Mexico.

Nasser, L. (1991): Helping Students to Upgrade their Van Hiele Levels. *Proceedings of PME-15,* Vol. 1, p. lxxi, Assissi, Italy.

Nasser, L. (1992): *Using the van Hiele theory to improve secondary school geometry in Brazil.* Unpublished doctoral dissertation presented at the University of London, UK.

Nasser, L. & Sant'anna, N.F.P. (1995): Long term effects of a geometry course based on the van Hiele theory. *Proceedings of PME-19*, Vol. 1, p. 213, Recife, Brasil

Santos, V.M.P. & Nasser,L. (1995): Teachers' awareness of the process of change. *Proceedings of PME-19,* Vol. 2, pp. 186-193, Recife, Brasil.

Tall, D. (1995): Cognitive growth in elementary and advancedmathematical thinking. *Proceedings of PME-19*, Vol 1, pp. 61-75, Recife, Brasil

Usiskin, Z. & Senk,S. (1990): *Evaluating a test of van Hiele levels: a response to Crowley and Wilson.* JRME, 21, pp. 242-245.

Van Hiele, P.M. (1986): *Structure and Insight.*. Orlando: Academic Press.

SOLVING WORD PROBLEMS WITH DIFFERENT MEDIATORS: HOW DO DEAF CHILDREN PERFORM?

Terezinha Nunes and Constanza Moreno
Institute of Education, University of London[1]

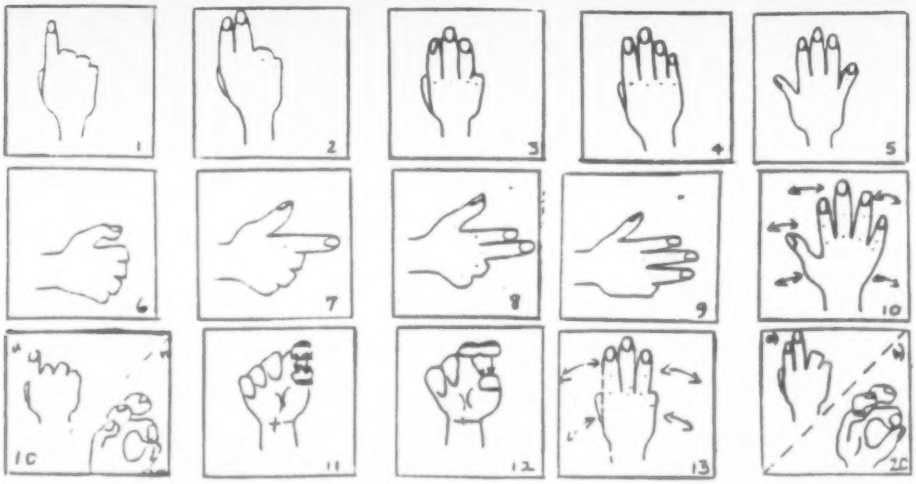
We investigated the acquisition of the signed algorithm by six profoundly deaf primary school children and the effect of different mediators - objects versus the signed algorithm - on their word problem solving performance. Similarly to the acquisition of the written algorithm by hearing children, deaf children's calculation errors with the signed algorithm could be attributed to the structure of the numeration system operated on. Results of the problem solving tasks indicated that the children performed significantly better when using objects than when using the signed algorithm, and that this difference could be explained by the formalization involved in the use of the algorithm. Level of problem difficulty followed the same pattern documented for hearing children.

Several studies (e.g. Nunes, Schliemann, and Carraher, 1993) demonstrate that the type of system of sign used to mediate reasoning in mathematics significantly influences subjects' problem solving performance. Thus it is important to analyse a variety of systems of signs which are used in school in the teaching of number concepts. We investigated the use of signed numbers (see Figure 1) as mediators for deaf children's development of additive strutures. Signed numbers are used by some children in a procedure we will refer to as "signed algorithm". The algorithm involves simultaneously signing each of the numbers in an addition or subtraction sum with a different hand - for example, one hand signs 8 while the other signs 7 when the pupil wants to solve 8 + 7. Increments of one are then added to 8 (the value to be operated on) at the same time as 7 (the value of the transformation) is progressively decreased by one. The result is achieved when the hand signing the transformation reaches 0; it will be read on the other hand, which works as the notepad. In a subtraction problem, the minuend is signed with one hand, the subtrahend with the other, and they are both decreased by one until the subtrahend reaches 0.

---

[1] This project was supported by grants from the Institute of Education and the Child Development and Learning.

Figure 1: The signed numbers used by the children in this study



The use of signed numbers has hardly been investigated so far. A notable exception is the work of Secada (1984). When he matched hearing and deaf children for age, deaf children lagged behind their hearing counterparts in the acquisition of counting. However, when they were matched with respect to their counting range, Secada found no difference between the hearing and the deaf children's use of counting to answer questions that required the use of counting. In spite of the similarity in performance, the two groups of children made different sorts of errors, which were closely related to the counting system they were learning. Deaf children's errors reflected the double-base of their system, which changes rules at 5 and 10, whereas hearing children did not make the same sort of mistake.

When hearing children use their fingers during problem solving, their fingers are representations of the objects; in this case, it is possible to perform on the fingers actions which do not correspond to arithmetic operations - for example, to count from *a* to *b* to solve a missing addend problem or to compare 4 and 5 fingers by setting them into spatial correspondence. As Marton and Neumann (1990) have pointed out, children can solve some addition and subtraction problems using their fingers before they can indicate which operation would be needed to solve the problem. In contrast, when deaf children use their fingers in the signed algorithm, the fingers represent numbers rather than objects, and a formal representation of the solution is required; the operation to be performed must be decided from the outset. We were thus led to predict that the deaf children would perform significantly better in problem solving if they were allowed to use objects to solve the problems than if they had to rely on the signed algorithm. Their errors reflect the need to formalize the problem solution rather than lack of understanding of the problem situation.

Due to the paucity of studies on the signed algorithm, we had several aims in our study: 1) to describe the difficulties in mastering the signed algorithm; 2) to contrast the children's performance in two types of problem solving condition, one where they relied on the school-learned procedure to calculate solutions and a second condition where they could use objects to solve the problems; and 3) to describe the level of difficulty of different problem types.

Method

Subjects: Six profoundly deaf primary school children (age range 6 to 8 years; mean age 7.23), users of Sign Supported English, attending a London primary school for the deaf participated in this study. Only one child had deaf parents.

Design: The study was conducted in two phases. In the first phase, the children were first video-taped in the classroom during six mathematics lessons taught by a specialist teacher. The main aim of the lessons was to teach the signed algorithm. During each lesson, two children were focused on, yielding a total of two sets of observation per child. These records were used to describe the difficulties of acquiring the signed algorithm.

In the second phase, the children were interviewed individually in three sessions. In the first session, the children answered addition and subtraction sums. Those sums which were solved correctly by at least five of the six children were chosen for use in the word problems. This choice of easy sums allowed for a better analysis of the impact of the means of representation on problem solving because we could elimintate the possibility that failure in a word problem resulted simply from not knowing how to carry out the sums. In the second and third sessions, the children solved a total of 16 addition and subtraction word problems presented in random order. In session two the children could use manipulatives (cut out figures representing the objects) to solve the problems whereas in session three they were asked to use the signed algorithm.

Considering that changing mediators within a session was not practical, a fixed order for the conditions of testing was used with objects as mediators in the first session because this choice would not favour our expectation of better performance with objects than with the signed algorithm. The sessions were about one week apart.

Procedure: The sums in session one were presented both in Signed English by an expert signer (the second author) and written in arithmetical notation on cards; the children signed the result. The word problems were presented in Signed English, and repeated as often as needed to ensure that the children understood the situation.

Results

1. Difficulties in mastering the signed algorithm

Although the description of the signed algorithm suggests a straightforward process, our analysis of the video-tapes of the children's learning processes indicated that the calculation procedure is complex. The four types of error identified can be traced to the operations on the mediating signs.

First, the children need to distinguish fingers as countable objects from fingers as signs for numbers. When manipulating countable objects in a subtraction, for example, it does not matter which objects are removed first. When using the finger algorithm, it does matter which fingers are retracted in which order. For example, when counting down from 8, if instead of retracting the middle finger the children retract their thumb, they go from 8 to 2 instead of 7 (see Figure 1).

Second, the children need to become experts in counting-down with their fingers, a necessary skill for both addition and subtraction in signed algorithm. Three types of count-down errors were observed.

a) Failure to "carry the five" may occur when children count down from numbers above 5. If they forget to carry the five, a confusion between 5 and 0 results.

b) Failure to "carry the ten" may happen when numbers above ten are involved in calculation. There are two common ways of signing numbers in the teens (either by successive signs indicating one-zero and the number of units or by signing the number of units while shaking the hand) and both involve some memory processes. Failure to "carry the ten" results in confusing, for example, 14 and 4 or 13 and 3. Failure to "carry the five" and "carry the ten" were observed in five of the six children.

c) "Skipping 5" may be observed when the children count down from a number above 6. The thumb must be retracted twice in succession, once going from 6 to 5, and then again counting down from 5 to 4. This type of error was only observed in the most inexperienced child and even in her case it was not frequent.

A third difficulty of the algorithm relates to the need to distinguish the number operated on (signed by the hand that works as the pad) from the value of the transformation being carried out (signed by the active hand). If at any point during the calculation process the children forget which is which, they reverse the progressive transformations, obtaining the wrong answer. This error was observed in two children.

A fourth source of difficulty was related to a specific teaching choice made by the teacher. Her aim was to get the children to realise that it is more efficient to use the larger addend as the number to be operated on irrespective of where it appears in the sum. She thus inverted the order of the addends when the smaller one appeared first and some children seemed to become confused. This inversion in the order of addends requires that the children understand the commutativity of addition. Although the most skilful child in the group had no difficulty in changing the order of the addends, all others made some errors when the order of addends was changed.

In short, the signed algorithm cannot be learned by the simple copying of gestures. Children need to understand the algorithm and master the difficulties related to the structure of the counting system. However, we wish to point out that the signed algorithm also has advantages: it allows the deaf children to mediate and control their computation processes rather than leaving computation to the mercy of their ability to recall verbal number facts, an advantage that was documented by Moreno (1994).

2. Differences across testing conditions

The children's performance in the two problem solving conditions differed significantly. The mean number of correct responses for the problems solved with objects was 9.45 whereas the mean number for the problems solved with the signed algorithm was 1.98 ($p < .001$ for a two-tailed t-test for correlated samples). This result supports the hypothesis that different mediators affect children's reasoning during problem solving.

Further analysis was carried out in order to investigate whether it was the need to formalize the route to problem solving when using the signed algorithm which led to the decrease in the children's performance. If this is the case, one would expect the difference between the two problem solving conditions to be significant only when the actions carried out with objects to solve the problems do not correspond to the formalization in terms of arithmetic operations, yielding a pattern of non-significant differences for direct addition and subtraction problems and significant differences for the other sorts of problem used in the study.

The predicted pattern was observed in a general fashion. There was no difference in the children's performance when the direct problems were considered: only one child made a mistake in one of the three direct problems using the signed algorithm and another made one mistake with the objects. In contrast, significant differences were observed when the actions carried out with objects to solve the

80

problems do not correspond to the formalization in terms of arithmetic operations. In the equalize problems, the children performed rather well in the object condition (mean = 3.6 correct over 4 problems) but rather poorly in the signed algorithm condition (mean = 1); the difference was significant at the .012 level (two-tailed t-test for correlated samples). A significant difference was observed also for missing subtrahend problems, which children solve with objects by building a set, counting out the objects which are said to have remained, and then counting out the objects which "were lost" in the story: the mean number of correct responses was 1.7 (in two problems) in the objects condition and 0.2 in the signed algorithm condition, a difference that was significant at the .001 level (according to a two-tailed t-test for correlated samples). The inverse problems (missing addend and missing minuend) were rather difficult for the children in both conditions: the mean number of correct responses was 0.2 (in three problems) for the objects condition and 0.1 for the signed algorithm condition. Because of the floor effect, no significant difference can be found.

In short, the comparison between the children's performance in the two conditions indicates that the different mediators affect the children's reasoning. This finding is significant for the education of deaf children because their performance was rather good in word problems when they could use their informal reasoning strategies. This result contrasts with previous findings that indicated that deaf children are poor solvers of word problems. Perhaps the reason for their failure in word problems in other studies results from an interaction of the problem presentation (problems were presented only in English whereas we used Signed English) and the need for the children to use written procedures, which, like the signed algorithm, are based on the formalization through the choice of an arithmetic operation.

3. Pattern of development in solving additive reasoning problems

The preceding comparisons indicated that deaf children's progression through word problem solving is clearly similar to that observed among hearing children:

direct and equalize problems are easiest, followed by missing subtrahend, and then inverse problems when objects are available. Comparison problems were not included in this study because Moreno (1994) had observed a floor effect for deaf children of this age level. This result suggests that it is the need to perform more operations of thought in solving problems that increases their difficulty rather than the particulars of the linguistic form used in the word problem, because Signed English uses different resources in the presentation of the same problems to children than does English.

Conclusions

This study demonstrates in a new way that the systems of signs used during problem solving affect reasoning. Deaf children's errors with the signed algorithm can be traced directly to the structure of the counting system and the algorithm used, just as the errors observed among hearing children can be traced to the numeration system and the mechanics of the written algorithm. Further, the same children solving problems with different sorts of representational support perform significantly differently. This shows that it is not the children's deficiency of comprehension but their difficulty with formalization that results in poor performance when solving certain classes of problems. Finally, the results indicate that deaf children's progression through the conceptual field of additive transformations parallels that of hearing children, even if they might, for a variety of reasons, lag behind.

REFERENCES

Marton, F., & Neuman, D. (1990). Constructivism, phenomenology, and the origin of arithmetic skills. In L.P. Steffe & T. Wood (Eds.), *Transforming young children's mathematics education. International perspectives* (pp. 62-75).

Moreno, C. (1994). The implementation of a realistic mathematics education programme in a special school for the deaf. Unpublished MSc Dissertation, Institute of Education, University of London, London.

Nunes, T., Schliemann, A.D., & Carraher, D.W. (1993). *Street mathematics and school mathematics*. New York: Cambridge University Press.

Secada, W. G. (1984). Counting in sign: The number string, accuracy and use. Unpublished Doctoral dissertation, Northwestern University, Evanston, Chicago.

TELLING DEFINITIONS AND CONDITIONS: AN ETHNOMETHODOLOGICAL STUDY OF SOCIOMATHEMATICAL ACTIVITY IN CLASSROOM INTERACTION

MINORU OHTANI
Faculty of Education, Kanazawa University, JAPAN

This ethnomethodological study investigates how sociomathematical activity in a seventh-grade classroom is constructed and sustained. The following discussion consists of two parts. First part involves an indication of the theoretical framework and the justification for the methodology used. In the framework, sociological constructs, such as participation rights and politics of representations are taken as objects for analysis in which mathematical constructs, such as definition and condition play methodological units. The second part involves description and interpretation of episodes that occurred during a year long classroom participant observation. Analysis of transcripts of records reveal that telling definitions and introducing conditions function as social resources widely used to sustain privileged participation rights and to dominate particular mode of representation.

INTRODUCTION

The investigation has been influenced by the theoretical underpinnings of ethnomethodology (Garfinkel, 1969; Leiter, 1980). Central to its formulations is the notion that reality is implicitly but deliberately constructed by member of a social group. Ethnomethodology investigates member's methods of creating and using sociological constructs, patterns of social interaction such as the famous triadic "initiation-reply-evaluation" (Mehan, 1979), and member's accounting practice to attain the factual character of the social reality (Ohtani, 1983).

In mathematics educational research, ethnomethodological and related interactionist analysis of mathematics classroom detected several specific patterns of social interaction such as the "elicitation pattern" (Bauersfeld, 1994); "funnel pattern", "staging pattern", and "thematic pattern" (Voigt, 1995). These research showed how teacher and students elaborate and use these patterns in the course of social interaction. Resent research envision mathematical aspects of interaction such as the "sociomathematical norms" (Yackel, 1993; Voigt, 1995) and "mathematical rationale" (Kumagai, 1994).

This article focuses on another aspects of "sociomathematical" interaction. The following discussion consists of two parts. First part involves an indication of the theoretical framework and the justification for the methodology used. In the framework, sociological constructs, such as participation rights and politics of representations are taken to the study of classroom mathematical activity in which mathematical constructs, such as definition and condition play analytical units. The second part involves description and interpretation of an episode that occurred during a year long participant observation in a Japanese seventh-grade classroom.

THEORETICAL FRAMEWORK

ASPECTS OF SOCIOMATHEMATICAL ACTIVITY

Resent research (Bauersfeld, 1994; Cobb, Wood, Yackel, & McNeil, 1992; Lampert, 1990) have a common and persuasive vision of mathematics classroom as socioculturally mediated milieu. Different classroom cultures mediate different values with respect to classroom interaction, and with respect to mathematical activity. In everyday classroom practice, teacher and students coordinate the extent to which they participate in a particular mathematical activity. The members role in participating mathematical activity, the extent to which they take direct responsibility, and innovation of certain discourse type varies across classroom practices. In this regard, activity in mathematics classroom is referred to "sociomathematical". In this study, two sociological constructs which are related with sociomathematical activity are taken as objects for study. These involve "participation rights" and "politics of representations".

PARTICIPATION RIGHTS

Sociological studies on classroom organization illustrate some systematic patterns in teacher student interaction. These interactional patterns are seen as patterns of distribution of participation rights allocated for teacher and students. Teacher's action such as individual nomination, invitation to bid, and invitation to reply are seen as natural methods of transferring rights to the students. And, to raise ones hand is seen to get his/her rights to innovate speech. Likewise, interrupting another speech means authoritarian act of invading ones rights. In this regard, classroom sociomathematical activity can be characterized as patterns of distributions and assertions of rights to participate and accomplish mathematical practice. Such distribution pattern of rights is called "mathematical participation structure" (Ohtani, 1994a).

POLITICS OF REPRESENTATION

Events and objects are vague and ambiguous. The choice of a particular way of representing events gives them a particular meaning. There is often a competition over the correct, appropriate or performed way of representing objects and events. Proponents of various positions in conflicts waged in and through discourse attempt to capture or dominate modes of representation. The competition over the meaning of ambiguous events and objects in the world has been called the "politics of representation" (Mehan, 1993).

In educational research, a similar competition among representations were investigated: decision of learning disabled child (Mehan, 1993); reading (Minick, 1993); science teaching (Wertch & Toma, 1995); mathematics teaching (Ohtani, 1993). These studies commonly illustrate that one mode of representing the world gains primacy over others, consequently a hierarchy among modes of representations is formed. It is tenable that a similar competition over the meaning of events and objects is played out in everyday mathematics classroom discourse.

DEFINITION AND CONDITION AS ANALYTICAL UNITS

In search of theoretical constructs that will provide analytical units to investigate participation structure and politics of representation, I shall draw on the works of Lakatos (1976) and Wieder(1974).

As Lampert (1990) observed, Lakatos portrays historical debates within mathematics about what a proof of a theorem represents by constructing a conversation among a group of students that contains mixed within it many axiological belief system, conceptual horizon among mathematicians over last several centuries. In the conversations, Lakatos demonstrated how new knowledge develops in the discipline with proof following a zig-zag path starting from conjectures and moving to the examination of premises through the use of counter examples or refutations. In the midst of an argumentation, revised definitions and conditions are progressively introduced in light of refutations. It seems that formation and revision of definitions and introducing conditions are indispensable and essential components that constitutes mathematical activity.

However, relative differences between social and psychological life of mathematicians and students need further consideration of aspects of formulating definitions and introducing conditions in institutional settings. In this regard, Wieder's study on "convict code" in a halfway house (Wieder, 1974) gave insight for the development of analytical units. In his participant observation, Wieder found that convict code not only serves as moral order that residents should observe but also serve as interpretive cognitive framework for staff to explain residents' deviant behavior. Further, he depicted that convict code operates as a devise for urging or defeating a proposed course of action, a devise for legitimately declining a suggestion or order.

DESCRIPTION AND INTERPRETATION OF EPISODE

DATA COLLECTION

The episode I shall examine here comes from a seventh-grade classroom (with thirty two students) in a junior-highschool located at Tsukuba city, Japan. A year long participant observation starts April 1993, when the author enter the class as teaching assistant. Every lesson, three times a week, was audio-video taped for later analysis. These records are transcribed and utterances and nonverbal behavior are attributed to speakers and numbered for ease of reference.

DATA ANALYSIS METHODS

Microethnography or "constitutive ethnography" (Mehan, 1979) was adopted for data analysis strategy. This strategy is characterized as "structuring structures" (Mehan, 1979). The constitutive ethnography studies both recurrent patterns of interaction occurring in classroom and members' methods of structuring activity that results in these patterns. A comprehensive analysis of the entire corps of data generates a provisional scheme of interaction. The scheme is constantly confronted by discrepant cases until the researcher has derived a candidate recursive pattern. In order to insure that the recursive pattern or structure uncovered by the researcher converges with that of the participants, I adopt the research strategy that is analogous to "breaching study" (Garfinkel, 1967). This methodological procedure consists in observing participants actions where normal circumstances are disrupted. The interactional work become visible when normal circumstances are disrupted. In a disruption, people engage in recovery work to reestablish the normal patterns of interaction. This recovery work displays and informs what is normally hidden interactional work that accomplishes normal forms of interaction.

RESULTS

In the course of analyzing huge amount of data corpus, I found that mathematical definition and condition operate as social and multi-consequential devise for coordinate and sustain classroom social interaction.

PARTICIPATION RIGHTS

I shall outline an episode that will provide some illustrations of pattern of interaction with regard to mathematical conditions. In the beginning of a lesson, teacher assign a task to concerning to elaborate a convenient way to calculate mixed addition and subtraction of positive and negative integers. The teacher invited students to propose six integers arbitrary. In the following excerpt, the teacher and students are designated by T and SS, and individual student is designated by his/her initial.

#01 T : Hey, you. [points to KN]
#02 KN : Minus three.
#03 T : [writes the number] Next. [points to HN]
#04 HN : Minus three.
#05 T : [writes the number and points to MN]
#06 MN : Minus one.
#07 T : [writes the number and nods to KE]
#08 KE : Plus one.
#09 T : Huh? <2> [with a puzzled look] You're going to//
#10 SS : Yeah! Yeah! [many talking out]
#11 T : [points to HK]
#12 HK : Plus ten.
#13 SS : [giggling]

#14 T : How come (•••)? <2> [loudly] What! What's the matter? [points to OT]
#15 OT : Minus ten.
#16 SS : [laughing]
#17 T : [introduces operations: (-3) + (-3) − (-1) − (-1) − (+10) + (-10)]
#18 OT ; Damn you.
#19 HG : Damn you. Why? Malicious! Malicious!

In the transcripts, the teacher successively nominate students and the students propose integers. The students, however, successively propose quite simple and trivial integers whose absolute values are the same (#02, #04, #06, #08). The teacher implicitly suggests students that their proposals are irrelevant (#09). Students none the less welcome simple number combinations (#10) and propose further trivial integers (#12, #15). This means that normal circumstances are disrupted. In such a disruption, the teacher introduces arithmetical operations to reestablish normal patterns of interaction (#17). I observed that, in his recovery work, introducing conditions (arithmetical operations) serves as a means to defeat students ideas and proposals and to suggest appropriateness of integers student should propose.

POLITICS OF REPRESENTATION

The following episode will provide an illustration of politics of representation with regard to mathematical definitions. The students are working on linear equation using balance beam. The problem is as follows.

> In a balance beam chocolates and candies are leveled.
> There are three chocolates on the left side and six candies
> on the right side. How many candies are leveled to one chocolate?

The teacher nominated a student (SH) and ask him to show his solution. SH proceeded to the board and manipulated concrete objects. He removed one chocolate from the left side and two candy from the right side and arrived at an equality of two chocolates to six candy. The episode begins as they start to work on the equality.

#01 T : There are six candies. Now we need the number of candies to a chocolate.
#02 It's easy to find that the answer is tree (candy), because there are six
#03 candies for two chocolate. Can you tell us about what you did? What did
#04 you do?
#05 SH : [no response]
#06 T : Look this. [pointing to a list of the equivalent transformation simplify-
#07 cation procedures] Which one did you use? This one [multiply both side]
#08 or that one [devise both side]? Which one? Divide both side .... or

#09 multiply?
#10 SH : Multiply.
#11 T : Multiply. What do you multiply both side. <2> To both side, what do you
#12 multiply?
#13 SH : Same number.
#14 T : Huh, the same number. Now, tell us what's that.
#15 SH : Two.
#16 T : Multiply by two? Um. <3> If you multiply two, what are the results.
#17 SH : Four (....) [in audible]
#18 T : Four X. <2> And this side.
#19 SH : (....) [in audible]
#20 T : What?
#21 OT : Twelve.
#22 T : It's twelve, isn't it. But, you see, equation is, <2> you see, this is a
#23 equation with unknown X. Okay? <2> In sum, to solve an equation, ...
#24 What we should do? A solution of an equation is <2> is transformed into
#25 "X equals to" <3> Shh! Guys shut up! [loudly] "X equals to something".
#26 Then, the solution must be in the form "X equals", or "one chocolate
#27 equals to." Now we have arrived that two chocolate equal to six candies.
#28 Isn't it a reverse procedure if you multiply two to the both side? <2>
#29 You see we have arrived that two chocolate equal to six candies. We
#30 should transform to "X equal". What do you do. What number to
#31 multiply?
#32 SH : X (....) [in audible]
#33 T : Multiply by X? <2> If multiply X by X, then what's the result?
#34 SH : [no response]
#35 T : To the power of?
#36 SH : (Two)
#37 T : X squared. It isn't correct. Tell us what number do you multiply?
#38 SH : [no response]
#39 T : Um, um. Okay, let me ask the question to somebody. Huh, [points to ST]
#40 ST, are you ready to join? Can you tell me what you did.

In the transcripts, the teacher expect the student SH to use equivalent transformation simplification method (#01-#04). The student SH, however, did not respond to the teacher (#05). The teacher again ask SH which equivalent property can be used (#06-#09). He replied wrong answer (#15). Here the expected interaction wad disrupted. The teacher implicitly suggested that his answer are not correct (#16). In such a disruption, the teacher told formal definition concerning a solution of an equation in order to reach expected solution (#22-#31). The student SH none the less present wrong answer (#32). The teacher eventually employed most powerful and common method that is to nominate excellent student to reach correct solution (#39-#40).

In this episode, it seems that the teacher and the student SH have been guided by different situation definitions. The teacher represent the situation in terms of mathematical notion of equation-solvig procedures such as the equality relation and system properties. The student SH, however, represent the situation in terms of vernacular and everyday knowledge of balance beam, chocolate, and candy. An inspection of the protocol shows that telling formal definition serves as a means of privileging decontextualized formal representation over contextualized vernacular representation of balance beam.

DISCUSSION

SOME FEATURES OF TELLING DEFINITION AND CONDITION

Analysis of the episodes shows that telling definitions and introducing conditions function as social resources widely used in order to sustain privileged participation structure and to negotiate certain representation of problem situation rather than cognitive resources used to analyze and describe problem situation and to construct mathematical dialogue.

Condition functions to regulate students' mathematical activity in ways that are appropriate for the classroom setting. And by introducing condition the student engages in a process sanctioned and regulated by the teacher. Condition becomes directive of imperative which he is expected to follow.

Definition functions as a strong implicit message that decontextualized mode of representation should be privileged, even if another mode could be used to describe an objects or event more appropriately and usefully in activity settings. Thus, in place of diversity or heterogeneity, telling mathematical definition designs to get the student to participate in formulating the problem in particular way.

In sum, telling mathematical definitions and conditions are much more a method of moral persuasion and justification that involve the following social functions: to sanction and defend unexpected or insignificant interaction with students; to defeat students ideas and proposals; to justify teachers control over students; to attain a degree of uniformity of what it transmits, and so on.

CONCLUDING REMARKS

The study showed that telling definition and condition create a social reality for students and showed some of the way in which telling definition and condition was persuasive and consequential. However, the investigation itself raised several additional questions to show how telling definition and condition was productive of a social world of real events and to show how talk could be heard as telling definition and condition. These questions require a closer look at experience of students than ethnographic reportage of it. It require a turn from a description of events and objects which were experienced by the ethnographer to a description of the course of experiencing those occurrences as events and objects.

REFERENCES

Bauersfeld, H. (1994). Theoretical perspectives on interaction in the mathematics classroom. In R. Biehler et al., (Eds.), *Didactics as a scientific discipline* (pp.133-146). Dordrecht: Kluwer Academic Publishers.

Cobb, P., Wood, T., Yackel, E., & McNeil, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal*, 573-604.

Garfinkel, H. (1967). *Studies in ethnomethodology*. Prentice Hall.

Kumagai, K. (1994). Mathematical rationales for students in the mathematics classroom. In Ponte, J. P. & Matos, J. F., (Eds.), *Proceedings of the eighteenth annual conference of the international group for the psychology of mathematics education*, 3 (pp.112-119). University of Lisbona.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27 (1), 29-63.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge University Press.

Leiter, K. (1980). *A primer on ethnomethodology*. New York: Oxford University Press.

Mehan, H. (1978). *Learning lessons: Social organization in the classroom*. Harvard University Press.

Mehan, H. (1993). Beneath the skin and between the ears: A case study in the politics of representation. In S. Chaiklin and J. Lave (Eds.), *Understanding practice: Perspectives on activity and context* (pp.241-268), Cambridge University Press.

Minick, N. (1993). Teacher's directives: The social construction of "literal meanings" and "real worlds" in classroom discourse. In Chaiklin S. & Lave J.,(Eds.), *Understanding practice: Perspectives on activity and context* (pp.343-374). Cambridge: Cambridge University Press.

Ohtani, M. (1993). Problem solver as a reality constructor: An ethnomethodological analysis of mathematical sense-making activity. In Hirabayashi I., Nohda, N., Shigematsu, K., & Lin, F-L., (Eds.), *Proceedings of the seventeenth annual conference of the international group for the psychology of mathematics education, 2* (pp. 254-261). University of Tsukuba.

Ohtani, M. (1994a). Social construction of "mathematical participation structure". *Bulletin of Institute of Education, University of Tsukuba, 18(1)*. (in Japanese)

Ohtani, M. (1994b). Sociocultural mediateness of mathematical activity: Analysis of "voices" in seventh-grade mathematics classroom. In Ponte, J. P. & Matos, J. F., (Eds.), *Proceedings of the eighteenth annual conference of the international group for the psychology of mathematics education, 3* (pp.384-391). University of Lisbona.

Voigt, J. (1995). Thematic patterns of Interaction and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), (1995). *The emergence of mathematical meaning: Interaction in classroom cultures* (pp.163-201). Hilsdale, NJ: Lawrence Erlbaum Associates.

Wertsch, J. V., & Toma, C. (1995). Discourse and learning in the classroom. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp.159-174). Hillsdale, NJ Lawrence Erlbaum Associates.

Wieder, D. L. (1974). *Language and social reality: The case of telling the convict code*. Mouton.

Yackel, E. (1993). Sociomathematical norms, argumentation and autonomy in mathematics. *Paper presented at the 1993 Annual Meeting of the American Educational Research As* [illegible] anta, GA.

MAKING SENSE OF CHILDREN'S PATTERNING

Jean Orton and Anthony Orton, University of Leeds, U.K.

Different types of pattern questions were given to over one thousand 10-13 year olds. Correlations between the different types - strings, cycles, tables, linear and quadratic - were found to be high. Information from individual interviews was used to inform the analysis. Stages, tracing progress through the linear questions, were suggested for the responses and were found to conform to a Guttman scale. Levels, based on qualitative considerations, were more appropriate for the quadratic questions and were compared with the levels of the SOLO taxonomy.

Rationale and previous research

Over recent years, patterning activities have become a feature of the mathematics curriculum, and attempts have been made to measure the effect and value of such experiences in quality and amount of learning. To what extent are children able to perceive, understand and use patterns in generalizing, in coming to terms with algebra, and in problem solving? As a result of their monitoring of mathematical performance from 1978 to 1982, the APU (undated) wrote that: "finding terms in number patterns gets progressively more difficult the further the terms are from those given in the question; more pupils can continue a pattern than can explain it; number pattern rules are described by a large proportion of pupils in relation to differences between terms; and generally, oral explanations of rules ... are given by more pupils than can write an explanation". Lee and Wheeler (1987) used both linear and quadratic patterns with students aged 15 to 16. They suggested that two kinds of students were successful, namely those "who hit upon a usable pattern perception and pushed it through", and those "that were flexible in pattern perception and could see a new pattern when one was unproductive". Stacey (1989) used linear generalizing problems with students aged 9 to 13. As well as documenting student methods she reported that "the constant difference property was widely recognized and could be used by a large majority" to move from one term to the next, and also that it was common to find students who moved "from a correct linear model to a direct proportion model for harder parts of a question". Pegg (1992) reported on three studies, for all of which the major innovation was the adoption of the SOLO taxonomy of Biggs and Collis (1982) as the theoretical perspective. Redden (1994) described a study based on linear generalizing problems in which, again, the SOLO taxonomy was used as a theoretical framework for interpreting the data. Our work sought to explore pupils' responses to a wide range of pattern questions. What patterns are noticed by pupils? To what extent can stages or levels be used to classify pupils' patterning ability?

Test items and responses

A written questionnaire was used with 1040 pupils from Years 6, 7 and 8 (ages 10-13). Thirty of the children were also given individual interviews and asked, for each task on the questionnaire, how they had obtained their answer. The conclusions discussed herein are based both on the written responses and on an analysis of the interview data. The final version of the questionnaire, adapted from that used in a pilot experiment (Orton and Orton, 1994) contained five types of questions, referred to subsequently as strings, cycles, tables, linear and quadratic. In this paper the question numbers in [ ] brackets indicate position on the questionnaire.

Strings, Cycles and Tables

There were five strings, for all of which students were requested to "fill in the next number", for example [1] 1 3 5 7 and [4] 1 2 4 8 16. Performance was generally good on strings, yet [4] proved to be at least as difficult as the cycles questions. The pupils' responses clearly indicated that differencing was automatic, and for some was the only strategy available. Two examples of cycles ([6] and [15]) are provided here. The questions required pupils to "fill in the missing numbers (shapes, letters)", and facilities were high. The questions referred to as tables were all structurally quite different from each other, and again pupils were asked to "fill in the missing numbers". Question [12] is given as an example. The data obtained from the strings, cycles and tables were used in the subsequent analysis, but space precludes any detailed discussion of the results and the nature of the responses.

[6]

1	2	3	4
2	3		1
3	4	1	
	1	2	

[15]

a	b	c	d
b	c		a
c	d	a	
	a	b	

[12]

	2	4	6	8
2	4	8	2	6
4	8		4	2
6	2	4	6	
8		2	8	4

Linear

The two items in this category were based on linear functions. Each item included a number of questions based on a particular problem situation. Such items have been referred to as superitems in some of the literature (Collis, Romberg and Jurdak, 1986). The two problem situations were:

[16] A bicycle hire firm charges £2 per hour to hire bicycles
How much will it cost to hire a bicycle for ... hours ?

[17] Bobby has just got a new job selling encyclopedias.
His basic wage is £16 per week but he gets a bonus of £5 on top of this for every set of encyclopedias that he sells.
In a particular week he sells ... encyclopedias. How much was he paid ?

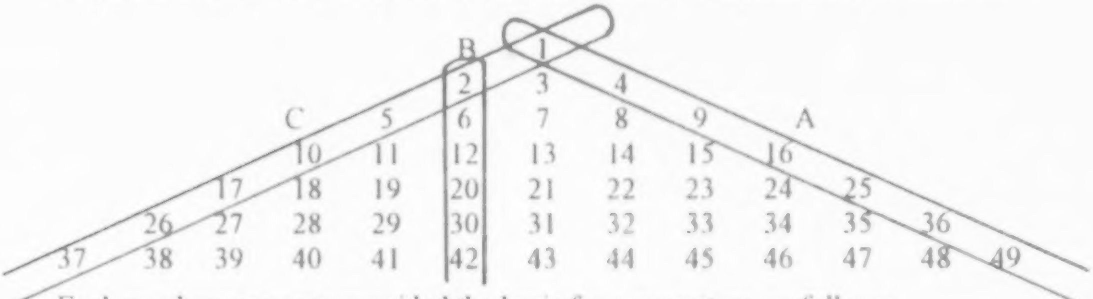
The questions required students to state the cost (wage) for 1, 2, 3, 4, 5, 6, hours (sets), for 20 hours (sets), for 100 and for n. Students were also asked to explain how to work out the cost from the number of hours (wage from the number of sets), before going on to deal with n hours (sets). The results are shown in Table 1.

Table 1 : Facility levels (percentages correct) for Linear Questions [16] and [17]

[16] / [17]	Next few	20th	100th	nth
Year 6	95.2 / 68.3	89.0 / 56.8	74.5 / 47.5	1.4 / 0.7
Year 7	95.7 / 69.3	89.1 / 52.3	79.2 / 41.7	10.4 / 7.0
Year 8	97.7 / 79.6	94.1 / 69.9	87.8 / 62.1	25.6 / 19.4

Quadratic

The quadratic questions, [9], [10] and [11], were all based on the three sequences, A, B and C respectively, marked on the triangular array of numbers:



Each number sequence provided the basis for a superitem, as follows:

> Write down three things you notice about the numbers in the loop.
> If there were more rows of numbers in the triangle, what number would come next in the loop?
> What would be the 20th number in the loop?
> What would be the 100th number in the loop? Can you see a simple rule for working it out, and if so, what is it?
> What would be the nth number in the loop?

In general, the quadratic questions proved very difficult for the children. About half of the pupils managed to give the next number in each loop, but the many arithmetical errors reduced the facilities for the 20th and 100th numbers, and few could cope with the nth term. Over half of all pupils omitted the question about a simple rule for Loop A and this rose to about 80 per cent for Loop C. As in Question [17], facilities for Year 6 pupils were sometimes higher than for Year 7, but these differences were not significant. The written responses to what was noticed in each loop show that many children focused on the differences between successive terms or described what was noticed in terms of addition. The alternate pattern of odd and even numbers in Loops A and C was also widely noticed.

Table 2 : Facility levels for the Loop A / B / C Quadratic questions

A / B / C	Next	20th	100th	nth
Year 6	46.9 / 45.9 / 45.5	4.1 / 3.4 / 0.0	4.1 / 0.7 / 0.0	0.0 / 0.0 / 0.0
Year 7	45.7 / 52.0 / 36.6	16.2 / 3.8 / 0.5	12.4 / 1.8 / 0.2	4.1 / 0.7 / 0.2
Year 8	64.4 / 61.3 / 45.9	28.2 / 14.0 / 2.3	21.3 / 8.8 / 0.9	15.4 / 7.2 /2.5

Correlations

As well as a total **Score** for the whole questionnaire, five separate scores were obtained for **Strings, Cycles, Tables, Linear** and **Quadratic**. A matrix was then obtained, showing correlations between these six scores and **Age**:

	Age	**Strings**	**Cycles**	**Tables**	**Linear**	**Quadratic**
Strings	0.242					
Cycles	0.099	0.260				
Tables	0.099	0.270	0.369			
Linear	0.230	0.420	0.328	0.332		
Quadratic	0.236	0.364	0.268	0.314	0.467	
Score	0.248	0.568	0.612	0.722	0.761	0.702

[All correlations were highly significant ($p < 0.002$, $n = 930$)
Data was not used for pupils who did not complete the paper.]

Although some correlations are more significant than others, it is enough to record that these values confirm relationships between all pairs of scores. There are no surprises here. Basically, children who are competent on one type of pattern are also competent on the other types.

It was decided to pursue the analysis further, and to use factor analysis to see if it exposed any useful further information. However, the Kaiser-Meyer-Olkin measure of sampling adequacy proved to be too low (Norusis, 1990), at 0.293. By removing the variable **Score** all the remaining variables had suitable measures of sampling adequacy (close to 0.8), and a high value was obtained for the Kaiser-Meyer-Olkin statistic. The procedure was continued with a Principle Components analysis. Only one factor produced an eigenvalue greater than 1.00 (41.3% of the variance), and thus Varimax Rotation was not applicable. This factor appears to represent overall patterning ability, perhaps not surprising, with the highest loading on **Linear**. Thus the factor analysis did not provide much additional information, but nor did it produce any unexpected or embarrassing surprises.

Stages and Levels

Several attempts were made to investigate stages in the development of patterning ability. An earlier study reported in Orton and Orton (1994), had revealed that the capabilities of adults answering quadratic pattern questions in an examination situation could be classified in stages. The four questions which made up the superitem in that earlier study involved stating the next, 10th, 50th and nth term in a sequence based on dot patterns. If a student was able to answer a particular question they were always able to answer all the previous questions, and thus the stages of development were:

Stage	0	1	2	3	4
	No progress	Next term provided	Next and 10th terms	Next, 10th and 50th terms	Next, 10th, 50th and nth

This was a deceptively simple classification which proved difficult to apply in the present study. It was not always the case that a correct 100th term had followed a correct 20th term and it was difficult to know what to do with pupils who provided a satisfactory response to the nth term but who had not provided a correct 20th or 100th term or both. This led to subdivisions of the stages to cover all the possibilities, using the letters t, x and y to indicate errors in the next, 20th or 100th terms respectively. Thus a child who gave the correct 100th term but the wrong 20th was classified as at Stage 3x, while a pupil who provided the correct 20th but not the next term was at Stage 2t. Other subdivisions were used at Stage 4:

Stage 4a	A correct verbal statement,
Stage 4b	A creditable attempt at an algebraic expression,
Stage 4c	A correct algebraic representation, but not necessarily the simplest.

The classifications in Tables 3a and 3b nevertheless do provide some information about the stages that pupils had reached on the two linear questions. Progression through the three years is self-evident. The percentages for Stage 4c indicate that a small but growing number of pupils are able to present an adequate algebraic representation. And the percentages of pupils within Stage 4 reveal quite rapid and continuous progress in the ability to describe verbally or algebraically or both in Question [16], and significant progress between years 7 and 8 in Question [17]. The evidence we have confirms that a verbal description is often available to pupils when an algebraic statement is not.

Table 3a: Percentages of pupils reaching particular stages in Linear Question [16]

Stage	4c	4b	4a	4	3	2	1	0
Year 6	1.4	0.0	54.5	55.9	23.4	11.7	6.2	2.8
Year 7	8.5	1.9	57.0	67.4	16.1	8.7	5.9	1.9
Year 8	21.6	4.0	53.8	79.4	11.7	4.0	3.8	1.2

Table 3b: Percentages of pupils reaching particular stages in Linear Question [17]

Stage	4c	4b	4a	4	3	2	1	0
Year 6	0.7	0.0	23.0	23.7	23.7	14.4	21.6	16.5
Year 7	5.0	2.0	21.6	28.6	17.6	8.8	27.1	17.8
Year 8	14.4	4.8	26.3	45.5	21.0	7.8	17.4	8.3

The fact that responses were often difficult to classify in Stages, for such reasons as an incorrect 100th term but a correct nth term, raises the question of whether our procedure was legitimate. The Guttman (1941) coefficient of reproducibility provides a measure of whether such digressions from a strict hierarchy are critical. Applying this analysis to our original data indicates that the responses can be classified in a true Guttman scale. (For [16] R=0.98 and for [17] R=0.96.) However, there has been considerable discussion of the legitimacy of this procedure (for example Moser and Kalton, 1971). Chilton (1969) discusses different error counting procedures and for the present study it was decided to use the method of Loevinger which Chilton describes to give the maximum score for errors. Chi-square procedures were also used to reject the possibility that the questions were independent. Moser and Kalton (1971), however, have suggested that all facilities should fall between 20% and 80%, which suggests some uncertainty about our conclusion.

Attempts to use a similar procedure with the Quadratic Questions were less satisfactory, because so few pupils were able to progress beyond Stage 1 in questions [10] and [11]. Thus a different procedure was adopted in which pupils' overall responses were placed at one of five levels:

Level 0	No progress at all,
Level 1	Pupil notices some properties of the numbers, with perhaps partial patterns described,
Level 2	Pupil notices a pattern but this is not described so as to allow the next number(s) to be derived,
Level 3	Pupil knows how to derive the next number(s) using patterns extrapolated from the differences,
Level 4	Pupil shows clear evidence of understanding the relationship, though an algebraic formula may not be articulated.

Thus for Question [9], 'some numbers are odd' and '16 and 36 are in the four times table' would be at Level 1; 'odd even odd even ...' and 'differences are odd' would be at Level 2; 'add 3, 5, 7, 9, 11, 13 and so on' and 'keep adding the next odd number' would be at Level 3; and 'times the row number by itself' and '1x1, 2x2, 3x3, ...' would be at Level 4.

In the pilot test paper pupils had been asked, "What can you say about the numbers that are inside the loop", and many had confined their response to one observation. In order to obtain more information about what was actually noticed the wording for the main study test was changed to, "Write down three things you notice about the numbers in the loop". This produced a wider range of observations, but then it was necessary for pupils to select which of their observations to use in the calculation of further terms. Individual interviews revealed the dangers of us wrongly assuming which method pupils were using to work out the next term. For example, some pupils noticed that the differences increased by 2 but used a counting-on procedure based on an extension of the triangle to work out the next term. Others mentioned (in Question [9]), 'It goes up by 3, 5, 7, ...' but then focused on a different pattern (16, 36, 56, ...) and gave 56 as the next term (in fact 8 per cent of all pupils answered 56). Individual interviews confirmed that some pupils who mentioned, for Question [9] or [11], that the differences go up by consecutive odd numbers were not aware that this pattern would continue for further numbers in the loop. For this reason it was decided that Level 3 should not be awarded to pupils who had spotted the difference pattern unless they were also either able to give correctly the next term in the loop or had given sufficient evidence of the application of their difference method. It was recognized that there was a risk of allocating only Level 2 to some pupils who had applied the difference method but made numerical errors, and that there was also a risk of allocating Level 3 to pupils who had observed the difference pattern, not realizing that it could be extended, but had used some other method to achieve the correct answer.

Some responses to the questions revealed improvement, that is a progression of levels through the three questions, although this could have been because of better use of language and not necessarily the development of understanding. There was also some indication of other pupils becoming tired of writing down observations. Certainly, for one reason or another, the number who left the observation section blank increased from 10 per cent in Question [9] to 30 per cent in Question [11]. Using this classification system, the results in Tables 4a, 4b and 4c were obtained for the Quadratic Questions. Many pupils achieved some success with Question [9], though it is startling that the number of pupils who did not recognize the square numbers was so high. Combining the facilities for Levels 3 and 4 gives a good indication of how well pupils could cope with interpreting the number patterns, and clear progression between Years 7 and 8 is evident.

Table 4a: Percentages of pupils classified at particular levels in Question [9]

Level	4 and 3	4	3	2	1	0
Year 6	30.6	6.8	23.8	25.9	31.3	12.2
Year 7	38.5	20.3	18.2	21.6	28.2	11.7
Year 8	53.7	29.1	24.6	24.8	13.9	7.6

Table 4b: Percentages of pupils classified at particular levels in Question [10]

Level	4 and 3	4	3	2	1	0
Year 6	24.7	0.7	24.0	40.4	20.5	14.4
Year 7	37.6	5.4	32.2	35.8	14.9	11.7
Year 8	47.1	13.1	34.2	31.9	7.9	12.9

Table 4c: Percentages of pupils classified at particular levels in Quadratic [11]

Level	4 and 3	4	3	2	1	0
Year 6	25.5	0.0	25.5	24.1	18.6	31.7
Year 7	25.7	0.7	25.0	24.5	20.0	29.8
Year 8	35.0	2.7	32.3	22.3	11.6	31.1

It is not appropriate to attempt to use the Guttman coefficient of reproducibility here because decisions about levels are not made on the basis of dichotomy. Responses are not simply right or wrong, but have to be judged in terms of quality. Instead, it is possible to relate our procedure to the SOLO model for the development of intellectual functioning. Working within the concrete operational stage, the match is good but not exact. There is clear justification for regarding Level 4 as the 'relational' level of the SOLO taxonomy, Level 3 as 'multistructural', Level 1 as 'unistructural' and Level 0 as 'pre-structural'. Level 2, however, appears to fall between multistructural and unistructural, and therefore appears to suggest the need for an intermediate 'partial-' or 'semi-multistructural' level in order to cater for considerable numbers of responses.

There will always be difficulties in attempting to classify children's levels of thinking. Written responses can be illuminated by oral explanation but there is the possibility of misinterpretation of any answer, oral or written, and some pupils seem uncertain of what is required and may fail to respond at all. Nevertheless some form of classification helps to clarify the overall picture and it is hoped that our attempts will enlighten the development of better research instruments.

References

APU (undated) *Mathematical Development: A Review of Monitoring in Mathematics 1978 to 1982*. Slough: NFER.

Biggs, J.B. and Collis, K.F. (1982) *Evaluating the quality of learning: The SOLO taxonomy*. New York: Academic Press.

Chilton, R.J. (1969) 'A review and comparison of simple statistical tests for scalogram analysis'. *Sociology*, **34** (i), 238-245.

Collis, K.F., Romberg, T.A. and Jurdak, M.E. (1986) 'A technique for assessing mathematical problem-solving ability'. *Journal for Research in Mathematics Education*, **17** (3), 206-221.

Guttman, L. (1941) 'The quantification of a class of attributes: A theory and method for scale construction'. In P. Horst (ed.) *The Prediction of Personal Adjustment*, 319-348. New York: Social Science Research Council.

Lee, L. and Wheeler, D. (1987) *Algebraic Thinking in High School Students: Their Conceptions of Generalization and Justification*. Montreal: Concordia Univ.

Moser, C.A. and Kalton, G. (1971) *Survey Methods in Social Investigation*. London: Heinemann Educational.

Norusis, M.J. (1990) *SPSS Statistical Data Analysis*. Chicago: SPSS Inc.

Orton, A. and Orton, J. (1994) 'Students' perception and use of pattern and generalization'. In J.P. da Ponte and J.F. Matos (eds) *Proceedings of the Eighteenth Conference for PME, Vol. III*, 407-414. Lisbon: Univ. of Lisbon.

Pegg, J. (1992) 'Research directions in school algebra'. In A. Baturo and T. Cooper (eds) *New Directions in Algebra Education*. Red Hill, Australia: Centre for Mathematics and Science Education, Queensland University of Technology.

Redden, E. (1994) 'Alternative pathways in the transition from arithmetic thinking to algebraic thinking'. In J.P da Ponte and J.F. Matos (eds) *Proceedings of the Eighteenth Conference for PME, Vol. IV*, 89-96. Lisbon: Univ. of Lisbon.

Stacey, K. (1989) 'Finding and using patterns in linear generalizing problems'. *Educational Studies in Mathematics*, **20**, 147-164.

CHILDREN'S INTUITIVE UNDERSTANDING OF AREA MEASUREMENT

Lynne Outhred and Michael Mitchelmore

School of Education, Macquarie University

This paper describes the strategies that young children used to solve rectangular area measurement tasks before being formally taught about area. These strategies are classified into five levels which appear to be developmental. Three crucial principles which children seemed to learn are that the rectangle must be covered without gaps or overlap; that it may be covered by a tessellation of squares; and that the number of units along each side can be found by measuring the linear dimensions. The implications for teaching length and area measurement are then addressed.

Despite being one of the most widely taught measurement concepts in the primary school, the concept of area is misunderstood by many children in the age range of seven to eleven years (Bell, Hughes, & Rogers, 1975). There is also a considerable body of evidence that suggests area is not well understood at secondary school (Clements & Ellerton, 1995; Bell, Costello & Küchemann, 1983). For example, Foxman, Ruddock, Joffe, Mason, Mitchell, & Sexton (1983) found that only 55% of eleven-year-olds in the UK could find the number of unit squares that would cover a T shape, given the lengths of its sides. When the term "area" was used the percentage of students who were successful dropped to 25%. Students were more successful when a square grid was shown superimposed on the shape than when they were given side lengths, presumably because the grid emphasised the covering aspect of area and because they could simply count the squares.

Prospective elementary teachers' knowledge of area has also been reported as inadequate. Simon and Blume (1994) indicated that although the student teachers in their study responded to area problems by multiplying, their choice of operation was often the result of having learned a procedure or formula for the area of a rectangle rather than the result of a solid conceptual link between their understandings of the relationship between side length and area. In a study by Tierney, Boyd and Davis (1990) many student teachers were observed to: confuse area and perimeter; apply the formula for finding the area of a rectangle to plane figures other than rectangles; consider area as "length x width"; use linear rather than square units; and equate changes in linear dimensions to changes in area (for example, many students believed that if the lengths of the sides of a square were doubled so would be its area).

There is general agreement that children's difficulties with rectangular area concepts are due, in part, to an emphasis on the formula and that if children do not understand the significance of partitioning regions into unit squares, any attempt to teach procedures to calculate areas will at best be learned by rote (Carpenter, Coburn, Reys & Wilson, 1975). To overcome the problem of rote application of formulae, concrete materials have been widely recommended as the basis on which to build abstract concepts. However, a number of research studies have indicated that children do not grasp the relationship between different forms of representation of mathematical ideas, in particular, between concrete activities involving covering rectangular figures and the formula for rectangular area

(Dickson, 1989; Hart, 1987, 1993). Moreover, the materials used to measure areas may affect children's thinking. Doig, Cheeseman & Lindsey (1995) in a study of eight-year-old children found that children who used wooden tiles to cover a surface were twice as successful as children who used paper tiles, because using wooden tiles avoided many potential problems such as overlap and gaps. Moreover, children may be able to find the number of unit squares that cover a shape but not realise that they are calculating the area (Bell, Costello, & Küchemann, 1983).

Children's drawings give further insights into their understanding of basic area concepts. In a small study of Year 6 and 7 children in Jamaica, Mitchelmore (1983) found that while most students gave the correct values for the areas of different shapes, none could draw unit squares in a given size rectangle and the children still had difficulties after practising covering and tiling activities. The conceptual problem would not be apparent using concrete materials, for the materials may so structure the array that children do not relate its construction to the dimensions of the rectangle. However, drawing tessellations of squares in a rectangle may focus children's attention on the relationship between the number of area units that cover a rectangle and the lengths of its sides, because the children have to determine how many units will fit along the adjacent sides.

Linking linear dimensions to rectangular area was identified by Simon and Blume (1994) as being difficult for student teachers. These authors suggested that learners need to understand the area of a rectangle as a quantification of surface and to visualise it as measurable by a rectangular array of units. The student teachers had a sense of the structure of a rectangular array, and seemed able to think about the target quantity as a combination of rows of area units. What was needed, Simon and Blume inferred, was the opportunity to develop a sense of how the linear measures and the area unit determined the shape and size of the array.

The current paper describes an investigation of how young children intuitively relate the area of a rectangle to its linear dimensions, part of a larger study (Outhred, 1993). The research questions were:

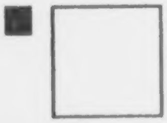
- What strategies do young children use to find rectangular areas before being taught the area formula?
- Can children's strategies be reliably classified into a sequence of developmental levels?

To provide answers to these questions, a large sample of children were observed as they solved a number of area measurement tasks.

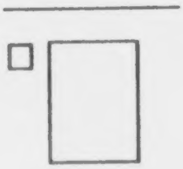
METHOD

The sample consisted of 115 children, with approximately equal numbers of boys and girls from a range of cultural groups, randomly selected from Years 1 to 4 in four schools in a medium socioeconomic area of Sydney. Children were presented with twelve array tasks, involving drawing, counting, and measurement skills; tasks were presented in a fixed order. The interviewer inferred children's strategies from a combination of observation and careful questioning as the children worked through the tasks. In this paper, only the measurement tasks will be discussed. Two measurement tasks (MT1 and MT2 below) were given to investigate children's understandings of the relation between side length and area. An additional measurement task (MT3) was given to 43 children as an extension activity.

The first task (MT1) was designed to tap basic concepts of covering using an informal unit. The unit (a 2 cm cardboard square) was provided and the children were asked to work out how many such squares would be needed to cover an 8 x 8 cm square shown on a sheet of paper. As the children were only given one unit they had to mark the rectangle in some way to work out how many congruent units would be needed to cover the figure. In this task the array could be constructed without "formal" measurement, for example, by repeatedly tracing the square.



MT1



MT2

The purpose of the second task (MT2) was to identify the strategies children used to find the number of standard units that would cover a rectangle using a ruler. The children were asked to work out how many 1 cm squares would be needed to cover a 6 x 5 cm rectangle. A square was shown to indicate the size of the units. Before they began this task the children were asked to measure the length of a 10 cm line to check their linear measurement skills. Success on task MT2 depended both on children's linear measurement skills and on their knowledge of the relation between linear measures and the structure of an array of squares.

A third task (MT3) was presented to children who measured at least one dimension on task MT2. This task was included to assess whether children who were able to determine the number of squares that would cover a rectangle could extend this concept to a task that involved the construction of more complex units. The task also served as a challenge for the children who could answer the core interview questions easily. The children were asked to work out how many 2 cm squares would be needed to cover an 8 x 10 cm rectangle when neither rectangle nor unit was shown.

RESULTS

In this section the strategies that children to solve each task are briefly described and then categorised into five developmental levels.

Classification of strategies for task MT1

1.1 Unsystematic estimation. In these strategies, children had no way of ensuring that they covered the region without gaps or overlapping. Thirty percent of children used such a strategy and none of them correctly found the number of units that would cover the square. For example, some children did not refer at all, or only made a visual reference, to the cardboard unit.

1.2 Moving and marking. The most common strategies seemed to be analogous to the physical action of covering the rectangle and involved systematically moving the square over the figure and marking its position in some way each time it was moved.

In some strategies, children marked either a side or a corner to keep the size of the unit constant. Twenty four percent of the sample used such strategies, of whom 62% correctly found the number of units. For the children who marked the endpoint of each unit move, this strategy was reasonably precise; errors usually resulted from adding extra rows or counting inaccurately.

The most common and accurate form of marking, however, was to trace the cardboard square repeatedly. Thirty four percent of the sample used this strategy, of whom 76% correctly found the number of units. This strategy was usually successfully executed, perhaps because (as when constructing the entire tessellation using cardboard squares) the material structured the array.

1.3 Informal measurement of side length. Twenty two percent of the sample used the given square to find how many units fitted along each side of the rectangle. Of these, 86% were successful in finding the number of units which covered the rectangle.

Classification of strategies for task MT2

In the second task the unit was shown pictorially but not supplied as a concrete unit. Therefore, the most commonly used strategies for MT1, moving and marking, could not be used. Thus, this task should be more effective in showing understandings of area than the more concrete task. About a fifth of the sample did not know how to measure the length of the line and were not asked to complete the second part of the task.

2.1 Array not completed. In addition to the 21% of children who could not measure length, a further 16% of the sample could not use a ruler to work out how many squares would cover the shape. Although these children had measured the line they usually drew individual squares with little regard to the size of the unit shown and to covering the region.

2.2 Array estimation. The main characteristic of this category is that, while children constructed an array, they used no reliable method of determining unit size. Twenty three per cent of the sample used such strategies, but they were rarely accurate, only two children obtaining the correct answer. If children estimated units along adjacent sides, then drew the array, they would essentially relate area to side length in the same way as children who measured the sides. However, only one child appeared to do this.

2.3 Measurement of one dimension. In these strategies, children measured one side of the rectangle and estimated the other. Fourteen percent of the sample used such strategies but only one child was successful. The technique used by nearly all of the children who used these strategies was to place the ruler along the top of the rectangle, mark or count units along the ruler, then slide the ruler down (estimating the vertical distance) and repeat the procedure.

2.4 Measurement of both dimensions. These strategies were used by 26% of the sample and 85% of these children obtained the correct solution. They involved the measurement of adjacent sides of the rectangle by marking each unit and either drawing the array; drawing squares along the two sides; or working out the number of units from side marks. A few children measured each side and multiplied the lengths.

Classification of strategies for task MT3

The strategies used to solve task MT3 provided additional information about the factors that make area a difficult concept for children. This measurement task was the most difficult because neither rectangle nor unit was shown. Moreover, the unit was a two centimetre square so measurements had to be coordinated in each dimension. The most common strategies were to draw an array; to mark off length units along the sides; and to calculate the number of squares from the length units without drawing. For example, Kelly (Year 4) without drawing explained that "You could halve each length so it's 4x5", while Tarun (Year 4) said "Every 2 cm is 1 so 10 is 5 and 8 is 4, so it's 5x4."

Classification of strategies into levels

Consideration of the strategies that were observed for the three tasks suggest that there are five levels in children's responses to rectangular area measurement tasks.

Level 1: Inadequate covering. Children do not realise that the whole surface must be covered with equal units; so they leave gaps, overlap units, or do not keep unit size constant.

Level 2: Enactive covering. Children realise that the whole surface of the rectangle must be covered. However, they may have difficulty ensuring that the covering is systematic and that the units are congruent.

Level 3: Array constructed, units estimated. Children realise that a tessellation of units gives a systematic covering of the surface. However, they do not see the relation between the size of the array and the lengths of the sides of the rectangle.

Level 4: Array constructed, one length measured. Children see the relation between the size of the array and the length of one side of the rectangle, but because they focus on one dimension, either rows or columns, they have difficulty constructing an accurate array.

Level 5: Array constructed, both lengths measured. Children measure or mark off units on both sides of the rectangle. This step seems to be essential for children to generalise from a tessellation of standard units (e.g., square centimetres) to a complex unit tessellation (e.g., two centimetre squares).

This sequence is developmental in the sense that each level would appear to be a prerequisite for the next level. However, each child did not solve all the tasks at the same level. For example, some children who used an estimation or tracing strategy (level 3) for MT1 used a level 4 or 5 strategy for MT2 and MT3. We may surmise that, at any given point in time, a child can operate up to a certain level but that the actual level employed to solve a particular task will depend on the specific demands of that task.

The empirical data tend to support the developmental nature of the above sequence. For example, most Year 1 students could not attempt MT2 although many traced the number of units that would cover MT1. Year 2 children predominantly used an estimation strategy for MT1 but a number had progressed to measuring one side length in solving MT2. By Year 3 and 4, a considerable proportion of the sample (about a third of Year 3 and three quarters of Year 4) measured the lengths of adjacent sides of the rectangle in some way in order to find its area.

DISCUSSION

The above five-level classification of children's strategies incorporates three key principles of rectangular area measurement:

Principle 1: The rectangle must be covered by unit squares without overlaps or gaps.

Principle 2: The unit squares must be congruent and aligned in an array with the same number of units in each row (and column).

Principle 3: The number of units in each row and in each column can be determined from the lengths of the sides of the rectangle.

The role of multiplication appears not to be crucial to an informal understanding of rectangular area measurement, although it would of course be essential to an understanding of the area formula. We therefore list it here for completeness:

Principle 4: The number of units in a rectangular array is the product of the number of units in each row and in each column.

Previous research on rectangular area measurement has tended to concentrate on Principles 1 and 4. The other tasks included in the present study but not reported here have already thrown light on how children learn to construct the rectangular array structure in Principle 2 (Outhred & Mitchelmore, 1992). The contribution of the present paper is to emphasise the significance of Principle 3, which appears to have been completely neglected in the literature. It may seem self-evident to adults that the numbers in the array must depend on the measurements of the sides, but it is clearly not self-evident to children. Let us consider why this is the case, and how children might be helped to learn Principle 3.

The basic problem that children who have learned Principles 1 and 2 have to solve is, How many unit squares fit along each side of the rectangle? Only if the unit square is physically available can this question be answered directly (as in Strategy 1.3 above). Otherwise, children must first realise that the length of a side (in centimetres) specifies the number of 1 cm unit *lengths* that will fit along that side; and then, that this number determines the number of unit *squares* that will fit along the side. Children's errors in measuring both the single line and the sides of the rectangle in MT2 suggest that many children had a purely instrumental understanding of linear measurement. Common errors included measuring from one end of the ruler (instead of the zero mark); measuring from the 1 mark; and counting marks instead of spaces. Other children repeatedly shifted their ruler 1 cm at a time, reminiscent of Strategy 1.3. Clearly the principle of ruler use (if you put the zero mark against one end of the line, the number against the other end of the line gives the number of 1 cm spaces) demands quite a degree of relational understanding. But this level of understanding would seem to be essential if the length of a side is to be related to the number of unit squares which will fit along it, especially if the side of the unit square is not 1 cm.

The learning of Principle 3 of rectangular area measurement would therefore seem to be dependent on a relational understanding of linear measurement, and in particular the use of a standard ruler. The teaching implication is that measurement with a ruler should not be taught simply as a mechanical skill. The ruler could instead be seen as only one of a number of ways of solving the general problem, How can you find how many of some unit length fit along a given length? For example, children might construct and label their own, non-standard "rulers" (such as 1 m strings marked every 10 cm). It would also be valuable for children to draw accurately the tessellations they make with concrete materials. Such experiences would help children understand area measurement by providing a closer link to length measurement.

REFERENCES

Bell, A., Costello, J., & Küchemann, D. (1983). *Research on Learning and Teaching. Part A*. London: NFER - Nelson.

Bell, D., Hughes, R., & Rogers (1975). *Area, Weight, and Volume*. London: Nelson.

Carpenter, T., Coburn, T., Reys, R., & Wilson, J. (1975). Notes from National Assessment: basic concepts of area and volume. *The Arithmetic Teacher*, October, 501-507.

Clements, M. A., & Ellerton, N. (1995). Assessing the effectiveness of paper-and-pencil tests for school mathematics. In B. Atweh & S. Flavel (Ed.), *Eighteenth Annual Conference of the Mathematics Education Group of Australasia* (vol. 1, pp. 184-188). Darwin, NT: Mathematics Education Group of Australasia.

Dickson, L. (1989). The area of a rectangle. In K. Hart, D. Johnson, M. Brown, L. Dickson, & R. Clarkson (Eds.), *Children's Mathematical Frameworks 8-13: A Study of Classroom Teaching* UK: NFER - Nelson.

Doig, B., Cheeseman, J., & Lindsay, J. (1995). The medium is the message: measuring area with different media. In B. Atweh & S. Flavel (Ed.), *Eighteenth Annual Conference of the Mathematics Education Group of Australasia* (vol. 1, pp. 229-240). Darwin, NT: Mathematics Education Group of Australasia.

Foxman, D., Ruddock, G., Joffe, L., Mason, K., Mitchell, P., & Sexton, B. (1983). *Mathematical Development*. Slough: National Foundation for Educational Research.

Hart, K. (1987). Practical work and formalisation: too great a gap. In J. Bergeron, N. Herscovics, & C. Kieren (Eds.), *Proceedings of the Eleventh International*

Conference on the Psychology of Mathematics Education, Montreal, Canada (vol. 2, pp. 408-415).

Hart, K. (1993). Confidence in success. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. Lin (Ed.), *Seventeenth International Conference on the Psychology of Mathematics Education* (vol. 1, pp. 17-31). Tsukuba, Japan: Program Committee.

Mitchelmore, M. (1983). Children's learning of geometry: Report of a co-operative research project. *Caribbean Journal of Education, 10*, 179-228.

Outhred, L. (1993). *The development of area concepts in young children.* Unpublished PhD thesis. Macquarie University.

Outhred, L., & Mitchelmore, M. (1992). Representation of area: a pictorial perspective. In W. Geeslin & K. Graham (Ed.), *Proceedings of the Sixteenth International Conference on the Psychology of Mathematics Education* (vol. 3, pp. 3-11). Durham, NT: Program Committee.

Simon, M., & Blume, G. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*, 472-494.

Tierney, C., Boyd, C., & Davis, G. (1990). Prospective primary teachers' conceptions of area. In G. Booker, P. Cobb, & T. Mendicuti (Ed.), *Proceedings of the Fourteenth International Conference on the Psychology of Mathematics Education*, 2 (pp. 307-314). Oaxtepec, Mexico: Program Committee.

RESPONSIVENESS: A KEY ASPECT OF SPATIAL PROBLEM SOLVING

Kay Owens

University of Western Sydney, Macarthur

Responsiveness emerged as a key aspect of spatial problem solving during an analysis of qualitative data gathered on primary school students engaged in spatial problem solving. Responsiveness may be regarded as a composite result of various cognitive processes including selective attention and affective processing as illustrated by an example. Theoretical and methodological issues are also discussed.

Background

The study described in this paper is essentially an investigation into the effects on primary school children of a program of spatial learning experiences. Typically spatial training studies have involved secondary or tertiary students in programs based on difficult two-dimensional and three-dimensional tasks (summaries in Eliot, 1987; Lean, 1984; Owens, 1990 & 1993). The studies by Del Grande (1987), Flores (1995), Lewis (1995), and Smith and Schroeder (1979) are exceptions but used mainly quantitative analyses. The qualitative study now reported explored *how young children learned* through spatial problem-solving experiences. The study involved the same learning experiences that had been shown to improve spatial thinking processes of primary school students (Owens, 1992b, 1993a & 1993b). The series of ten sessions--based on activities with pentominoes, tangrams, pattern blocks, and matchstick designs--were suitable for Year 2 and 4 students and encouraged some analysis and imagery.

Some studies into the thought processes of primary school children have used interviews but only a small number of students, concepts, or contexts have been involved (Mansfield & Scott, 1990; Wheatley & Cobb, 1990). The present study involved a large number of students, many concepts (ranging from shape concepts to angles and symmetry and other shape attributes) and 12 different classrooms.

Methodology

Spatial thinking is a mental activity and, as such, is not easily studied. The review of the literature indicated that retrospective comment was a reasonable approach for studying cognitive processing (Owens, 1990) by adults. This kind of data, however, needed to be supplemented with observational records when young students were involved. Further, the students participated in the activities in classrooms and so special procedures were needed to carry out a qualitative study. Several concerns were specifically met:

1. Controls for achieving reliability and validity with data were assisted by (a) purposefully selecting experiences, groups, and samples and (b) carefully recording, categorising, and making counts of described variables.

2. Observations were interpreted in terms of the theoretical position of the researcher and research literature (particularly on visual imagery, Goldin, 1987,

Osborne & Wittrock, 1983) but this did not limit the importance of unexpected results. Categories and relationships which were conjectured were further tested and modified in new situations.

3. The use of novel problems with concrete materials generated better quality retrospection data. The use of small groups encouraged children to talk freely and comment during the activities and during video-replay.

4. To some extent observation supported and was supported by verbal reports, but both methods of data collection provided their own forms of data.

5. Counts of examples of categories and connections between categories formed the basis of conjectures arising from patterns of behaviour or cognitive processes.

The use of observational and interview techniques for assessing thinking seems to be important and pertinent considering the recent emphasis on alternative assessment procedures in mathematics education and on drawing inferences from classroom discourse (Clarke, 1989; Stenmark, 1989).

Procedure for the Qualitative Study

The procedure involved selecting students in different samples, video-recording these students as they attempted to solve the spatial mathematics problem, describing incidents in the students' process of solving the problems, aligning these incidents with specified categories and subcategories, and following a grounded-theory approach to developing a "story-line" (Strauss & Corbin, 1990). A computer spreadsheet was used to record the descriptions of each incident and to record categories. These could be used for searching for patterns and for crosstabulations. The computer techniques were valuable in assisting the research process.

Several samples of students were selected from different situations (as recommended by Strauss & Corbin, 1990).

Stage 1. Data were obtained from two groups: (a) 52 adults (teacher-education students for whom the problems were novel) who commented on their thinking both as they worked through the problems and immediately afterwards; and (b) 13 children who worked alone on the spatial problems with minimal intervention from any other person and who commented on their thinking spontaneously or immediately afterwards. From this data, categories and subcategories were developed to form the basis for analysing the data obtained in the next stage (1994b).

Stage 2. There were two samples: (a) 12 students who worked in groups of three (two groups from Year 2 and two from Year 4), and who were asked how they were thinking immediately after working on each spatial problem (video-playback and oral reference to incidents were used to stimulate recall); and (b) 167 children (77 in Year 2 and 90 in Year 4) in six primary classrooms in Australia. The children were matched by school, year, class, and pretest score and randomly allocated to either a group in which students worked individually near others or a group in which students worked in small cooperative groups. Analysis of data led to the development of a model and descriptions of aspects of learning through problem solving. These descriptions suggested relationships between the categories of thinking processes which had been identified.

Stage 3. The "story-line" which was developed was checked by using two further samples: (a) 180 students in Grades 2 and 4 in Papua New Guinea who were placed into either the individual or group learning situation, and (b) 54 students (Years 2 and 4) from another Sydney region who were matched by Year and pre-test schore and randomly allocated to the different groups.

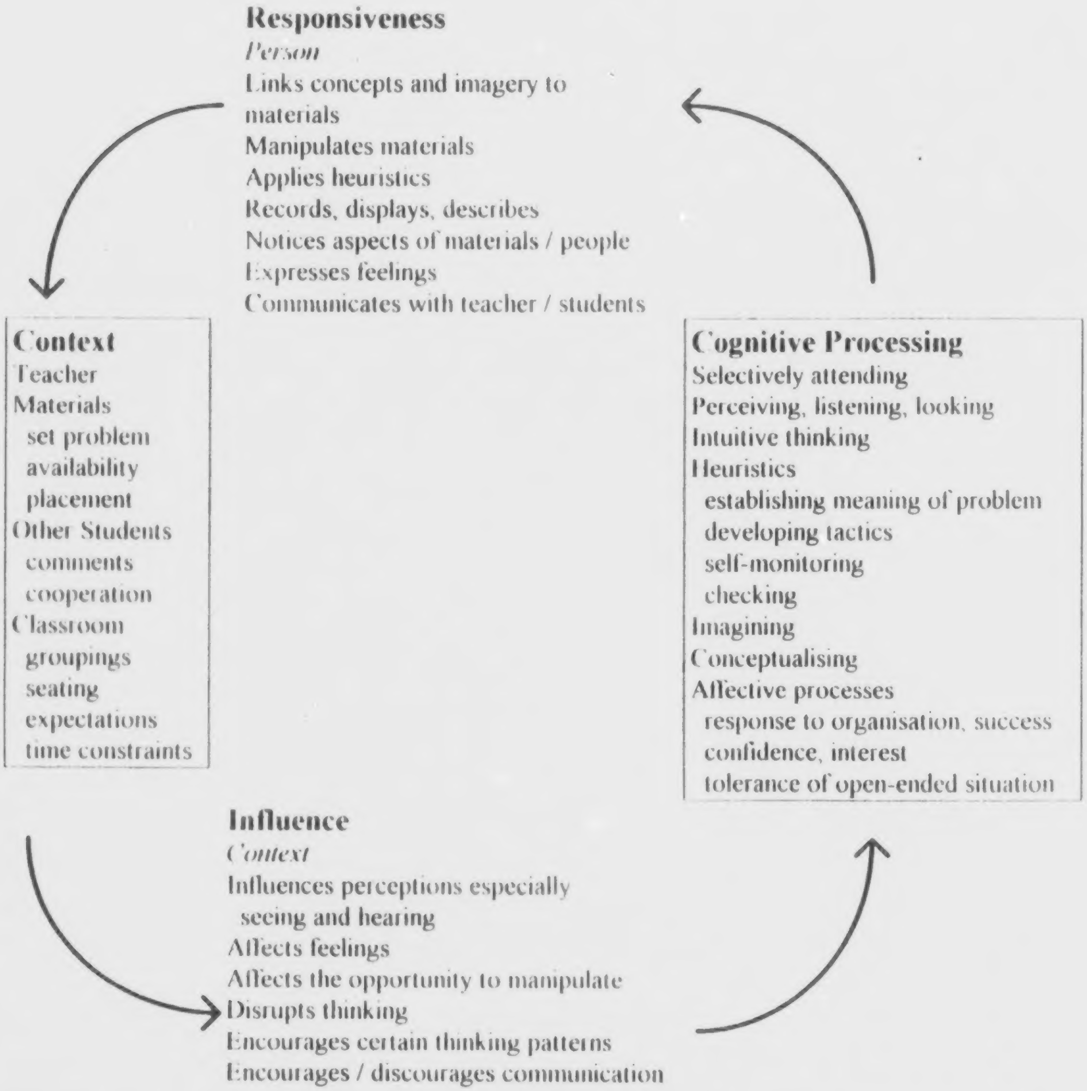
In stage 2 over 120 video-taped sessions were analysed. Each problem-solving session was considered as a series of incidents. An incident was defined as a relatively short period of time during which actions, expressions, or discourse changed as a result, it seemed, of changes in cognitive processing and interactions with people or materials. For each incident, students' spoken comments and observations of their actions and expressions were recorded and coded in terms of the defined categories and subcategories. The categories covered interactions between the students and the teacher, the use of concrete materials, and the cognitive processes suggested by the actions, words, and expressions of the students. These processes included affective processes (such as feelings), heuristics (such as tactics and decisions about progress in solving the problem), the use of concepts (such as what parts distinguish a particular shape), and the use of visual imagery (Owens, 1993, 1994a; Owens & Clements, in press).

Conjectures were made about how the various categories related to each other. From crosstabulations of subcategories, from "immersion" in the data, and from deliberate attempts to look for links and to check these links (Strauss & Corbin, 1990), patterns of relationships between subcategories were developed. These were illustrated extensively by examples (Owens, 1993). Four other researchers watched video-tapes and classified incidents in order to check and improve the classification process. The investigation generated a model which encapsulated students' thinking processes and their responsiveness during spatial tasks. The emphasis in this paper is on responsiveness -- a key composite variable which emerged from the data during the study.

Responsiveness

Students' responsiveness during active engagement in problem-solving activities is precipitated by their own thinking and feelings. Their responsiveness affects the immediate social and physical environment which, in turn, influences the person's thinking. It may, for example, be a change in position of concrete materials or a verbal reply by another student. This is illustrated in Figure 1.

Responsiveness implies a degree of understanding as well as involvement and interest in the activity. There is an ongoing dynamic relationship between students and their environment (that is to say, other students, the teacher, the classroom, and the task). Responsiveness embraces the kind of personal involvement and empathy which Fischbein (1987) associated with intuition, and Mason (1994) with "I-You" awareness in constructing concepts from "I-It" experiences.



Responsiveness
Person
Links concepts and imagery to materials
Manipulates materials
Applies heuristics
Records, displays, describes
Notices aspects of materials / people
Expresses feelings
Communicates with teacher / students

Context
Teacher
Materials
  set problem
  availability
  placement
Other Students
  comments
  cooperation
Classroom
  groupings
  seating
  expectations
  time constraints

Cognitive Processing
Selectively attending
Perceiving, listening, looking
Intuitive thinking
Heuristics
  establishing meaning of problem
  developing tactics
  self-monitoring
  checking
Imagining
Conceptualising
Affective processes
  response to organisation, success
  confidence, interest
  tolerance of open-ended situation

Influence
Context
Influences perceptions especially
  seeing and hearing
Affects feelings
Affects the opportunity to manipulate
Disrupts thinking
Encourages certain thinking patterns
Encourages / discourages communication

Figure 1. Aspects of problem solving

Changes in cognitive processing and in the learning environment occur throughout the period of a student's engagement in a learning experience. The student is continually perceiving, thinking and feeling, and then responding, and this dynamically affects the learning context. There is often a "snowballing" effect, not only on participation, but also on the extent and quality of imagery, concepts, understandings, and problem-solving tactics. The cyclical interaction pattern represented in Figure 1 provides for growth and continuity like a spiral, and for overlap of cycles like a double helix.

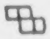
Cognitive processing embraces attending, perceiving, listening, looking, visual imagining, conceptualising, intuitive thinking and heuristic processing. Cognitive processing also incorporates affective processes such as reactions to the organisation of the classroom and to success, confidence, interest, and tolerance of open-ended problems. Points involving critical change in thinking are likely to involve both changes in affect and changes in understanding.

There is no hierarchy of cognitive processes suggested by the present model, and this is the case also in Goldin's (1987) model. Both models suggest that mental representations during problem solving arise from interactions between different non-hierarchical "languages" (Goldin, 1987). By contrast, Pirie and Kieren's (1991) model assumes there is a hierarchy in the development of students' understanding (from rich imagery to conceptualising, structuring and inventising) although their notion of "folding back" recognises the need to return from levels such as "formalised conceptualising" to earlier levels of thinking such as "image making." The four models (developed by Goldin, Owens, Osborne & Wittrock, and Pirie & Kieren) refer specifically to visual imagery. The roles of imagery and selective attention have been discussed elsewhere (Owens, 1994a, 1995; Owens & Clements, in press).

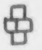
Cognitive Processing Influencing Responsiveness

An example will illustrate how responsiveness is often influenced by affect. In the second spatial activity James, a Year 2 student, was thoroughly involved in making new tetromino and then pentomino shapes from square breadclips. He also enjoyed commenting and in other ways expressing his achievements and feelings of pleasure. "Names" are used for different designs as illustrated and each paragraph (para.) is numbered for referencing in comments below.

1 James continues to count how many he has made, comparing his number with his friend's number.

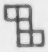


2 Using four squares, he makes a "Z," checks that it is all right and then makes a "cross" avoiding repeating the Z.

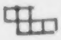


3 His friend points out "it is half here," so he changes it to a "T".

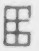
4 He begins with five squares, deliberately positioning the pieces to make a Z. Then he makes a "lineZ".

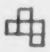


5 He notes his friend's shape saying "Yours has three columns, mine has two, she copied me." (Each made the lineZ in different orientations.)

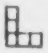


6 The teacher suggests that they work together but he keeps making shapes quickly and happily, commenting on how well he is going. He uses a tactic of beginning a new shape with "three-in-a-row." He counts his shapes and says "I'm beating her." He knows what he is making before he completes the shape, showing joy before he finishes making the shape. He places three-in-a-row, and claps as he makes a "C."



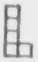


7 He cannot recognise the "odd" shape in different orientations despite moving his body to assist orientation. He changes the shapes to make the easily-recognised shapes "L3" and the "square-like shape," comparing the incomplete shapes with his short-term



memory images of those he has made (that is, he is not physically glancing at his shapes)

8 He changes his tactic from starting with three-in-a-row to beginning with four-in-a-row. He makes the "L4."



9 He quickly grabs the last five breadclips so that he can make another shape.

10 He wants to make a car but ends up with lineZ, globally deciding it is different and says "Oh, I can't make any more." His activity wanes when the teacher asks if they can find any shapes that are repeated in the group's work.

11 He recognises the repeated lineZ and L4.

There are several points to note about James' responsiveness. First, a friendly competition existed between the students and this motivated them to participate and achieve (paras 1 and 6). Certain affective characteristics are evident in his behaviour--his responses to his successes (paras 1 and 6), his competitiveness (paras 1, 6, and 9), his desire to make shapes (para. 9), and his loss of interest at the end (para. 10). Second, James' use of imagery influenced his responsiveness--not only his manipulation of materials (paras 3, 4, 7 and 10), but also his comment to his friend (para. 5) and his self-assessments (paras 6, 10, and 11) which tend to keep him on task. His imagery helps him to stay on task (paras 6 and 10). Third, he assessed or monitored his own progress on the task and this, too, influenced his responsiveness. He showed his monitoring by expressing how he was progressing (para. 1 and 6) and by changing his tactic in an appropriate way (paras 8 and 10). Finally, he expressed his understanding and knowledge (paras 3, 4, 5, and 11). The changes in his responses (paras 3 and 10/11) were precipitated by comments to him by his friend and by the teacher. Thus we see how his responsiveness was affected by (a) his understanding of the problem, (b) his use of visual imagery associated with comments by other students and the teacher, (c) his self-monitoring, and (d) his attitudes. At the same time, we can see how his visual imagery and tactics improved.

Mandler (1987) and McLeod (1993) have analysed the role of affect in learning and there can be no doubt that affect is an integral part of thinking influencing responsiveness. Positive feelings were strongly associated with deliberate manipulations (based on crosstabulations of categories), possibly because deliberate manipulations were likely to be associated with achieving results for the problem. Feelings were frequently associated with student self-monitoring. Much of this checking was achieved by placing pieces together or by looking and comparing pieces or configurations. The children expressed their feelings verbally on many occasions, and by attracting the teacher's attention, by clapping, and by smiling. Students working by themselves were generally positive about themselves working on the tasks, and in these cases interactions assisted the individual learners whereas the students working in groups were more inclined to show positive feelings about the task.

Students who experienced positive feelings about a task and themselves were motivated to continue to explore aspects of the task (cf. Goldin, 1988). These feelings were commonly associated with success as a result of deliberate manipulations, analytic conceptualising, and pattern and procedural imagery (Owens, 1995). Once students had developed a tactic, they usually worked well on a problem, and if they succeeded, they were then motivated to continue to try ideas and to keep on-task even if more advanced concepts and visualisations were needed.

In the orientation phase of problem solving a degree of uncertainty was tolerated, and negative feelings were usually mellowed by students making a start based on visual intuitive approaches, play, or interactions with others. A student occasionally might express negative feelings towards another student by making the task competitive or by disrupting productive work. Both positive and negative feelings were frequently manifested through interactions with others.

The teacher interacted in response to students' facial expressions, successes, uncertainty, and loss of interest. Students felt uneasy when the teacher questioned them. This may have been because the teacher was trying to help students with a difficult task such as generalising or disembedding parts of a shape. Alternatively, beliefs and established routines about mathematics classrooms (e.g., in some classrooms, no talking, no looking around, giving the answer that the teacher wants), seemed to influence interactions and to impact on the behaviour of students and the extent to which they were prepared to talk and share ideas with others.

In Summary

This paper has noted the way in which cognitive processing influences students' responsiveness which in turn influences the problem-solving environment and further learning. Responsiveness, although previously not recognised by mathematics education research, would appear to be an important variable influencing participation and performance in mathematics.

References

Clarke, D. (1989). *Assessment alternatives in mathematics.* MCTP. Canberra: CDC.

Del Grande, J. (1987). Spatial perception and primary geometry. In M. Lindquist & A. Shulte (Eds.), *Learning and teaching geometry, K - 12* (pp. 126-135). Reston, Va: NCTM.

Eliot, J. (1987). *Models of psychological space: Psychometric, developmental, and experimental approaches.* New York: Springer-Verlag.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach.* Dordrecht: Reidel.

Flores, A. (1995). Bilingual lessons in early-grades geometry. *Teaching Children Mathematics 1* (7), 420-424.

Goldin, G. (1987). Levels of language in mathematical problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 59-65) Hillsdale, NJ: Lawrence Erlbaum Associates.

Goldin, G. (1988) Affective representation and mathematical problem solving. In M. Behr, C. Lacampagne, & M. Wheeler (Eds.), *Proceedings of 10th Annual Meeting of PME-NA,* pp. 1-7.

Lewis, J. (1994). *An investigation into the strategies used by boys and girls when solving spatial problems.* MEd dissertation. Flinders University, South Australia.

McLeod, D B (1993) Research on affect in mathematics education: A reconceptualisation In D Grouws (Ed) *Handbook of Research on Mathematics Teaching and Learning*, (pp 575-596) New York Macmillan Publishing

Mandler, G (1989) Affect and learning Causes and consequences of emotional interactions In D B McLeod & V M Adams (Eds), *Affect and Mathematical Problem Solving: A New Perspective*, (pp 3-19) New York Springer-Verlag

Mansfield, H , & Scott, J (1990) Young children solving spatial problems In G Booker, P Cobb, & T N de Mendicuti (Eds), *Proceedings of the 14th PME conference* (Vol II, pp 275-282)

Mason J (1994) Researching from the inside in mathematics education Locating an I-you relationship In J P da Ponte & J F Matos, *Proceedings of PME18*, (Vol 1 , pp, 176-191)

Osborne, R J , & Wittrock, M C (1983) Learning science a generative process *Science Education, 67*, 489-508

Owens, K D (1990, July) *Getting inside the problem solver's head: using retrospection and observation to access spatial and problem thinking processes.* MERGA13 Mathematics Education Research Group of Australasia, Hobart

Owens, K D (1992a) Spatial mathematics a group test for primary school students In M Stephens & J Izard (Eds), *Reshaping assessment practices: Assessment in the mathematical sciences under challenge.* (pp 333-354) Melbourne Australian Council for Educational Research

Owens, K D (1992b) Spatial thinking takes shape through primary-school experiences In W Geeslin & K Graham (Eds), *Proceedings of the 16th PME Conference* (Vol 2, pp 202-209)

Owens, K D (1993a) Spatial thinking processes employed by primary school students engaged in mathematical problem solving Unpublished PhD thesis, Deakin University Press

Owens, K D (1993b) Factors pertinent to children's responsiveness in spatial problem-solving activities In E Southwell, B Perry, K Owens (Eds), *Space: The first and final frontier* MERGA15, Mathematics Education Research Group of Australasia, (pp 421-431)

Owens, K (1994a) Visual imagery employed by young students in spatial problem solving *In R. Killen (Ed.)* www swin AARE conf94 OWENK 161

Owens, K D (1994b) Spatial problem solving in mid-primary school Geometry WG PME

Owens, K D (1995, October) Responsiveness and selective attention Are they alternatives to cognitive load theory? ITEMS Seminar Series, University of Technology Sydney, Kuring-gai

Owens, K D & Clements, M A (in press) Representations used in spatial problem solving in the classroom, *Journal of Mathematical Behavior*.

Pirie, S , & Kieren, T (1991) Folding back Dynamics in the growth of mathematical understanding In F Furinghetti (Ed), *Proceedings of 15th PME Conference* (Vol 3, pp 169-176)

Smith, W S , & Schroeder, C K (1979) Instruction of fourth grade boys and girls on spatial visualization *Science Education, 63* (1), 63-66

Stenmark, J (1989) *Assessment alternatives in mathematics* Berkeley, Ca EQUALS Project, Lawrence Hall of Science

Strauss, A , & Corbin, J (1990) *Basics of qualitative research: Grounded theory procedures and techniques.* Newbury Park Sage Publications

Wheatley, G , & Cobb, P (1990) Analysis of young children's spatial constructions In L P Steffe & T Wood (Eds), *Transforming children's mathematics education* (pp 161-173) Hillsdale, NJ Lawrence Erlbaum Associates

Acknowledgement. The teachers for their cooperation, the students from whom I learnt, M A Clements for supervision and comments, L Outhred, P Fahy, J Conroy for second coding, University of Western Sydney for funding and support

THE FIRST ALGEBRAIC LEARNING
THE FAILURE OF SUCCESS

Mabel Panizza (*), Patricia Sadovsky (**) and Carmen Sessa (**)
(*) Ciclo Básico Común- (**) Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires

The objective of this research is to clarify the relationship between a teaching proposal - usual in our country - concerning the first algebra learning, and the meanings built by the students. The study is based on class observations made in a first year course of secondary school, which introduces the algebra by means of first degree equations with an unknown. Our analysis shows that the didactical proposal, on the one hand, emphasizes the continuity with arithmetics and, on the other hand, causes a fracture with the students' previous knowledge. Hypotheses about the meanings built by the students by means of this approach are posed. Some significant meanings are identified, which would be out or would be restrained by such ideas, in spite of the proposal success in relation to the teacher's goals. Moreover, it is pointed out that the observed system does not recognize as an objective to introduce the algebra through problems for which it is necessary.

1. Introduction

The purpose of our research is to identify the appropriation conditions of elementary algebra in secondary school students. Having set our interest in the **first** learning, we are essentially thinking about the use of letters as variables and unknowns.

Considering the theoretical and methodological frame of the **Theory of Situations** (Brousseau, G.; 1987), of **Didactics Engineering** (Artigue, M.; 1988) and of the **instrument-object Dialectics** (Douady, R.; 1986), in order to move forward in our research it is a priority to know better the relationships between the existing teaching proposals and the meanings the students get about the algebraic objects. Only by taking a certain distance, and defining the algebra teaching in our school today, as an object to be known, could we go beyond the position of extreme criticism that only shows the poor achievements of the students.

As many researchers have pointed it out (Cortés,A.-Vergnaud,G.-Kavafian,N.; 1990; Chevallard,Y.;1984), the algebra learning implies a significant epistemological break. From this perspective, we therefore planned to make a set of observations of introductory algebra lessons and to analyze the proposals of the text books that are important references for teachers.

The objective of this report is to clarify partly the system **supplies** and **demands** in the first learning of algebraic tools, the proposed **object approaches** and the assignments to be learnt in order to **be successful** in this matter. We basically intend to identify the meanings that are **left out,** even by learning well everything required by the system.

The work is based on observations made in a first year course of secondary school (12-13-year-old students). The selected school was a public one in the suburbs of the city. A school with a good reputation, where a work environment in the classrooms and among the teachers was better than the usual. The observed teacher - who offered to help with our research - has a good reputation, was able to keep the attention of the students, showed a good work with their mistakes, and considering the results of a final evaluation, to a great extent achieved the goals that she had set to herself. Whenever necessary, we will add the following to these observations: mentions to the used text book; data taken from a exploratory questionnaire made to a group of over one hundred 12 to 17-year-old children of another school; an interview made several months later to the class teacher and a clinical interview made to two girl students considered by the teacher to be among the best of the class.

2. Purposes of the System

The observed system teaches the beginning of the algebraic through the study of first degree equations with an unknown in seven lessons. All the assignments done by the students not only during the lessons, but also in a subsequent evaluation, were related to the achievement of these two purposes:

-To translate from the colloquial language into the symbolic one (that is to say, to put a problem or a word relation into equation).

-To solve equations.

The purpose of this paper is to show some **matters that remain dark** in spite of the **achievements of these objectives** and that could hinder the understanding of the different algebraic objects in the future.

3. The equation object at school

Equation is a definable notion in the field of the Mathematics Logic. Its precise definition, as a propositional function, does not seem to be within the 12-13-year-old students' possibilities that make their first approach to the use of letters as variables and unknowns. Y.Chevallard (1985) states that it is about a paramathematical notion, that is to say, a tool notion of the mathematical activity that is not normaily object of school teaching. Although the system is not intended for students to learn **"what is an equation?"**, in order for the students to acquire with meaning the different algebraic objects that the system does state as teaching objects (quadratic equations, linear equations with more than one variable, inequalities, functions), it will be necessary for the students to start building different approaches to the equation concept. This involves the elaboration of the root concepts, truth set, variable, equivalent equations, and so on.

In spite of the complex nature and of the difficult definition of the "equation" object, the subject of the first degree equations with an unknown is started with a general definition of equation. As a consequence of this, they appear overlapped - and at first without possibilities of differentiation for the 12-13-year-old student - two concepts of different nature: the equation concept and the first degree equation with a variable concept. The definition given to both concepts is that of **"equality with unknown"**.

We will not pause to analyze this identification between the equation notion and the first degree equation notion. We will try instead to describe in detail some problems that arise from the definition of equation as "equality with unknown".

What concepts do the students acquire from this presentation? It seems that the students deal simultaneously with two ideas of "equality with unknown". The first one is connected with the form of the expression and would be related to the presence of a sign or symbol that is not a numeral. According to the second idea, the students identify equality with numerical equality and unknown with "number to be revealed". They seem to believe that an equation is an equality among numbers where the x is "concealing" one number that takes part in the expression. The equation would therefore be a proposition - the statement of an equality - and not a propositional function. We believe that this idea is opposed to the equation idea as a restriction over a domain.

We have seen that when students are confronted to identifying equations from a list of expressions, they take the first of these ideas, acknowledging the difference between the "type" number and the "type" unknown. However, when trying to solve an equation without solution, they end up rejecting it as such, which leads us to think that they bring up to date the idea of numerical equality. The following part of the interview held with two students can be interpreted from this characterization: (The students had been suggested to solve the equation $3x + 2 = 3x + 8$. The students work correctly and arrive at the following expression: $3x - 3x = 8 - 2$) (*)

S1[1] Oh, oh, oh
I[2] What's the matter?
S1 If we put the unknowns of a member together it would be 3 x minus 3 x, we're out of x.
I So?
S2[3] Zero x
I Mm, and so?
S1 We can't solve it, we're out of x, unknown.
I And so?
S2 It's hardly an equation. Sure, it would be an equation it is an equation because it has unknown, but if you start solving it, it's not an equation because...
S1 (interrupting S2) It doesn't have equality
I Does it mean it may seem to me that something is an equation but later I may realize it is not? Can this be possible?
S2 Of course
S1 (first hesitates) Yes, because when you see something with unknown you say "oh! An equation!", but perhaps later when you solve it, it's not an equation because it doesn't have an equality, it doesn't reach a final result

……

S1 In my opinion, it is an equation, but you can't reach a final result, it doesn't have a solution.
S2 It is an equation because it has unknown, but it doesn't have equality. In my opinion, it doesn't have equality
S1 No, it doesn't have equality
I So, if the equation is an equality with unknown, what happens with it?
(S1 and S2 remain silent for a while)
I It is doubtful, isn't it?
S2 In my opinion, it is an equation because it has unknown, but it doesn't have equality.
I But you told me that an equation is an equality with unknown. So?
S2 (laughs) I mean, I don't know. At first sight it looks like an equation, but when you look at it carefully, it doesn't have equality
S1 No, when you start solving it, you can't reach a final result.
I And so?
S1 It's no longer an equation

---

[1] S1 stands for student 1
[2] I stands for interviewer
[3] S2 stands for student 2

I: That is to say, we said that all these are (a list of expressions that had been given to them to choose which ones were equations) Could it be possible for us to change our minds later?
S1: While you are solving it? Yes
I: In which case would you change your mind?
S1: In the case of an equation?
I: Yes
S1: If I reach a final result and it is an equality, it is an equation But if I can't reach a final result as in this case, in my opinion it's no longer an equation

In different moments of the interview, these two students seem to use the two ideas above mentioned. Let us analyze, for example, S2's phrase: "at first sight it looks like an equation, but when you look at it carefully, it doesn't have equality". In the first part of the sentence, the emphasis seems to be placed in the form of the expression, while at the end she gives priority to the idea of numerical equality. The same applies to the following sentence: "if I reach a final result and it is an equality, then it is an equation. But if I can't reach a final result as in this case, in my opinion it's no longer an equation", stated by S1. Summing up, the change of the expression status (the equation is no longer an equation) seems monitored by a change in the idea that is used and this one at the same time depends upon the type of assignment faced (to classify or solve). As far as the central assignments suggested by the teacher are those of translating and solving equations, the idea that the students most frequently use - and therefore the most acquired one - would be that of the **equation as numerical equality with number to be revealed.**

So far we have analyzed the students' interpretation of the definition "equality with unknown". Which is the teacher's perspective with regard to this matter? Is she aware of the distance between the equation as a propositional function and the ideas dealt with by the students? Which is her inner representation of the equation object? Is it different from the notion that she uses at the moment of teaching? Is she aware of those differences? If so, was she able to foresee some way of "negotiating" those differences?

Our observations do not let us answer completely the preceding questions. However, we can uphold that although the teacher has an inner representation of the equation as a formal object, not only the language used in class but also the suggested activities, aim to acquire the idea of the equation as numerical equality. Let us see some examples that explain our statement.

Firstly, the resource of the two-pan balance scale, which is used as a reference of the equations, is in agreement with the notion of unknown as number to be revealed. In effect, the equilibrium of the two-pan balance scale is much closer to the idea of equality already given than to a formal equality about which truth or falsity cannot be asserted. In any case, the two-pan balance scale model is interpreted by the students conveniently from the idea of equation as numerical equality.

Secondly, the teacher's speech about the teaching procedures to solve first degree equations would also acquire this idea. "If I add the same number to both members" - says the teacher - "equality is kept". To insist on "keeping equality" and to omit saying that the operations done keep the equality for the same values of the unknown, reinforces the idea that, from the beginning, the equation represents an equality among numbers.

All this analysis shows that the observed system intends to present the algebra as a natural continuity with arithmetics, concealing the true nature of the new objects. It is easy to foresee future difficulties in the students' development of new algebraic notions.

4. The absence of the notion of variable

We mentioned before that the idea of equation as equality among numbers is opposed to the equation idea as a restriction over a domain, which needs the notion of variable. This observation leads us to ask ourselves the following questions:
-Does the concept of the letters as unknowns cause any obstacles (according to Brousseau) to the idea of the notion of variable?
-Does the concept of the letters as variables contribute to the break process between arithmetics and algebra, break that is essential to the meaning of algebra?
-Is it possible to aim at the concept of the letters as variables before the students develop the notion of unknown? By means of which situations?

Although we are not able to answer the stated questions, some results taken through the exploratory questionnaire let us move further. In fact, hardly anybody was able to answer an item that requested to write a solution of the equation $3x + 2y = 7$. Some students "added" another linear equation and solved the system; and others answered that "it does not have solution". This result can be interpreted from something already noticed by many researchers, which is the strong devotion to the algorithms that the students develop during Mathematics learning. However, we also think that **the idea of equation as numerical equality might block the access to understanding the nature of two-variable equations, since in these ones it is impossible to interpret each variable as unknown number.**

A similar problem with quadratic equations with an unknown could be set forth.

Which is the approach that the observed system makes to the notion of variable? In the first lesson, the observed system starts by "defining" variable. In doing so, an example where it is requested to translate into symbols the expression " the perimeter of a square of side x" is appealed to. As from this point it is "shown" that in the formula $p = 4x$, "x does not have a fixed meaning and is, therefore, a variable". In contrast, it is defined that "a letter is constant if it always has the same meaning in a certain context". As a result of this and from the idea of equation as numerical equality with number to be revealed, **the symbol x in an equation would represent a constant!!**

On the other hand, it is important to make clear that, in spite of the visible concern of the system to show the difference between constants and variables, these notions were not dealt with again in any of the students' subsequent assignments.

5. A poor equivalence notion

The students deal with a restricted idea of the equations equivalence, in agreement with the idea of the equation as equality with unknown. Indeed, the operations allowed to solve an equation (to add, subtract, multiply or divide to both members) are,so because the underlying numerical equality is kept and not because equations with the same truth set are obtained.

As we have already noticed in the above item, the teacher's speech about "keeping equality" without stating that the truth set is actually kept, reinforces the students' ideas. Although the teacher would seem to deal with an implicit idea of equation as formal equality with an associated truth set, her language does not differentiate the formal equality references from the numerical equality references. What problems would such a restricted idea of allowed operations subsequently bring?

For instance, it would be "allowed to pass" from the equation $x + 1 = 2$ to the equation $x^2 + x = 2x$, since as far as x is a number, "equality has been kept" by multiplying both members by it. From this perspective it would be very difficult to understand the presence of a strange solution.

Another problem that we can foresee and that seems to be related to the substitution of the notion of "equivalent equations" by the notion of "keeping equality", appears in the linear equation systems when the students add or subtract two equations and they replace the two former ones by the one obtained. Of course, they have "kept equality" with this operation.

6. Break against continuity and fracture

So far we have set forth some hypotheses about the possible effects of the approaches that emphasize the continuity with arithmetics, on the subsequent learning concerning higher order equations, or linear systems. This "continuity layout" coexists in the observed system with a treatment that causes a fracture with the students' previous knowledge. This problem appears in the fact that in our country, the first algebra learning is at the beginning of secondary school. Unlike other concept breaks produced within the same institutional frame, the break implied by the algebra learning is inserted into another one, the institutional one (Chevallard,Y.;1985). Particularly, the students are exposed to changes coming from the differences between the inner representations that the primary school and secondary school teachers have of the knowledge to be taught, as regards mathematical objects and didactical objects.

Two observed important aspects - from our point of view - cause a fracture with the previous knowledge.

a) The first aspect refers to the status that the observed system provides to the symbolic treatment (in the algebraic sense) by identifying it with the mathematical treatment. In this way, all the activity previously done by the students is left out of mathematics, by not being considered symbolic. In that sense it is interesting to highlight the kind of answers obtained in the exploratory questionnaire from the question "Why do you think we use letters in mathematics?" A large number of answers were obtained, which do not grant any specific use to the letters, but a higher status, more complexity. The letters appear associated with the difficulty, to the extent where the mathematics class work becomes not understandable and alienated. Some typical answers were:
-"To reason more".
-"To make it more difficult, and therefore, to think more".
-"To understand more, but it is complicated".

b)The other aspect refers to the solution of equations. We have already pointed out that within the curriculum the equations first appear in the first year of secondary school. Some teachers, in order to prepare the students better for the "new institution", teach equations in the seventh (and last) grade of primary school. In this case, the difference between primary school and secondary school teachers in relation to this knowledge becomes important. The primary school teachers consider as a valid procedure "passing from one member to the other one". The official knowledge in secondary school, instead, takes as a valid procedure the one that comes from applying the uniform properties to both members of an equation (of the sum or the product). This of course is a problem, that becomes complicated because the old procedure is not known by all the students.

How does the system face it? According to the observations made, the students' previous knowledge is intended to be ignored, restraining its use in order to acquire "the

120

new" procedure. Such intent is supposed to beat a very strong inertia, since the students do not understand why they cannot appeal to the known procedure.

Likewise, once the canceling procedure has been taught, the secondary school teachers do not arrange the relevant actions so that the students can find that both procedures coincide (in the grounds that make them correct and in the effects they cause). In this way, not only an initial fracture with the previous knowledge is caused, but also the canceling procedure does not become a control element for the students concerning their habitual mistakes or hesitations in the "passing" procedure.

These types of approaches - that emphasize the continuity with arithmetics, and on the other hand cause a fracture with the previous knowledge - leave out a matter that we consider essential to the break negotiation that implies the passing from arithmetics to algebra: the dialectical game between the new knowledge and the old knowledge (Douady,R.;1986; Vergnaud,G.;1986 ; Chevallard,Y. ; 1984).

7. The need as a need

From the theoretical frame of the instrument-object Dialectics and the Theory of Situations, it is clear that in order to cause a break with arithmetics, the algebra must be presented as necessary. To find didactical proposals that satisfactorily solve the complex linkage between need and difficulty is a cause for worldwide research and also a future goal for our work.

On the contrary, according to the observations made, the purpose of the usual teaching in our country would not be to introduce the algebra as a tool that makes it possible to solve new problems. In fact, all the word problems set forth in class could have been solved using only arithmetical resources. When the teacher was inquired about this, she answered: "I think that it is better to learn the method in simple situations, in order to be able to apply it to more difficult situations later".
A.Cortés ,G. Vergnaud and N. Kavafian(1990) notice that while the assignments done by the students are from the beginning a response to the teacher's demand, their learning lies on the **acceptance of agreement**.

The observed teacher, to uphold this acceptance of agreement, explains several "reasons" in her lessons:
-"In the future difficult problems will appear, for which you will have to know equations".
-"When you have to solve problems with bigger numbers, you will need the equations".
-"If you do them mentally and you reach the wrong result, I will not be able to see where you went wrong". (expression that also leaves the teacher in charge of the control resource).

The other great aspect around which the system negotiates the problem of the need, seems to be based on reasons of "higher order": **the algebra appears as a way to make progress within mathematics, but this progress is due to criteria out of the student's needs.**

Bibliography

Artigue,M, (1988) Ingénierie didactique *Recherches en didactique des mathématiques*. 9 3, 281-308

Brousseau,G; (1987) Fondaments et méthodes de la didactique.*Recherches en didactique des mathématiques*. 7.2; 33-115.

Brousseau, G; (1983) Les obstacles epistemologiques et les problemes en mathématiques. *Recherches en didactique des mathématiques*. 4.2; 164-198.

Cortés,A, Vergnaud,G , Kavafian,N; (1990) From arithmetic to algebra: negotiating a jump in the lerning process. *Proceedings of the 14th PME Conference*, Mexico. Vol .2; 27-34

Chevallard,Y; (1984) Le passage de l'arithmetique a l'algebrique dans l'enseignement des mathematiques au college. Primiere partie. *Petit X 5*; 51-94

Chevallard,Y; (1985) La transposition didactique.*La pensee sauvage*.Grenoble.

Chevallard,Y; (1990) Le concept de rapport au savoir. Rapport personnel, rapport institutionnel, rapport officiel. *Irem d'Aix Marseillc*. Faculté des Sciences de Luminy.

Douady, R. (1986):Jeux de cadres et dialectique outil-objet. *Recherches en didactique des mathématiques*.7.2 ; 5-31.

Kieran, C y Filloy Yague,E (1989) El aprendizaje del álgebra escolar desde una perspectiva psicológica. *Enseñanza de las Ciencias*. 7.3; 229- 240

Sfard, A (1991) On the dual nature of mathematical conceptiones, *Ed. Studies in Math*.22.1; 1-35.

Sfard, A. and Linchevski,L Equations and inequalities.Processes without objects?. *The Hebrew Univ. of Jerusalem*.

Vergnaud,G; (1986) Long terme et court terme dans l'apprentissage de l'algebre. *Actes du premier colloque franco-allemand de didactique*.

EXTENDING THE EDUCATIONAL CONVERSATION: ADMINISTRATORS' VIEWS OF STAFF DEVELOPMENT*

Terry Peluso
Campbell Union High School District

Joanne Rossi Becker
Barbara J. Pence
San José State University

This paper reports on an interview study with six principals selected from 16 schools involved in a high school staff development project, "Building Bridges to Mathematics for All." In past research related to this project, we have used focus groups, quantitative data, and teacher interviews to ascertain the impact of the project on teacher beliefs and classroom behaviors. In this study, we interviewed selected principals to determine their views of the effect the project had had on their teachers' attitudes and pedagogy and on the mathematics curriculum at their schools. Results show that the principals felt the project was critical to implementing "algebra for all" and moving their schools toward reform in their mathematics curriculum.

Related Literature

Although in the 1986 edition of the Handbook of Research on Teaching, the chapter on mathematics education (Romberg & Carpenter) hardly mentioned research on teacher education, our knowledge of mathematics teacher development has progressed considerably in the last decade. One of the critical issues being studied relative to staff development for teachers is how one creates a milieu which fosters change in teacher behavior in the classroom. A number of researchers have highlighted the importance of teacher beliefs to change in teacher behavior (Cooney & Jones, 1988; Ernest, 1991). As late as 1988 Grouws pointed out that there was little information available about the overall design features of inservice education programs which produce changes in teacher beliefs and classroom practices. He called for careful studies which focus on the impact of various features of inservice education on classroom practices.

While connections between teachers' beliefs about mathematics and their classroom behavior have been made (Ernest, 1991) beliefs may also be influenced by other factors in the context of the school and the classroom. As Cooney (1993) has pointed out, there are several metaphorical ways of examining teacher beliefs. Considered in different ways, teacher beliefs might seem contradictory with

---

* This research was supported in part by a grant from the Eisenhower Mathematics and Science Education State Grant Program, California Postsecondary Education Commission, project # 758-7. The opinions expressed in this paper do not necessarily reflect the position of that organization.

classroom practice. This view of beliefs has informed our work as we have tried to ascertain not only what works to foster teacher change, but also what impedes it in the school setting.

Background

Since 1992 we have been engaged in an extensive staff development program with high school mathematics teachers, "Building Bridges to Mathematics for All." The program, associated with a national program entitled Equity 2000, focussed on new curriculum, updating pedagogy, and examining issues of equity. As part of Equity 2000 involvement, all of the schools in the two participating districts had committed to place all ninth grade students in a course at least as high as algebra 1/course 1 by Fall 1994. In addition to extensive staff development intended to reach all of the teachers, "Building Bridges" provided classroom coaching and purchase of mathematical materials such as graphing calculators and computer software to support instructional innovations in the classroom. The program is discussed in more detail in Peluso, Pence and Becker (1994).

In this and two other small-scale studies (Becker, Pence & Pors, 1995; Peluso, Pence & Becker, 1994) we have endeavored to evaluate the impact of the project on participant teachers. In particular, we have been interested in identifying specific aspects of the staff development which have helped to change teachers' beliefs about the teaching and learning of mathematics, and which have stimulated concomitant changes in classroom practices.

Methodology

In this study we conducted in-depth individual interviews with principals/vice principals of six of the 16 high schools that participated in the San José Equity 2000 Project. These principals were selected to represent schools of different size and ethnicity. The main purpose of the interviews was to determine principals' perceptions of the impact of the staff development on their teachers, particularly on teachers' attitudes and pedagogy, and on the mathematics curriculum being used in their school. The first author, who is not associated with the "Building Bridges" Project but is himself a high school principal, conducted all of the interviews. Interviews were not taped at the request of the subjects, but careful notes were taken and summaries were written immediately following the interviews.

Subjects. The schools ranged in size from 1200 to 2400 students, with a minority representation of 40% to 80%. All of these schools are comprehensive schools serving grades 9-12.

A total of eight administrators from the six schools were interviewed: the six principals and two vice principals. The administrator sample included six women and two men; seven were European-American and one was Hispanic.

Questions. The questions below were designed to elicit responses in areas of general information, content, pedagogy, and teacher/student attitudes with respect to the desired outcomes of the Equity 2000 project. A basic set of questions in each of these categories was asked of all subjects, with followup questions varying in order to clarify responses as necessary.

General Questions:

- Describe the school and student body.
- How many math faculty participated in the Equity 2000 institute?
- What changes, if any, have you noted in the way your math faculty has worked together over the past five years?
- How have project participants had the opportunity to experiment with new curriculum and pedagogy?
- Where did support for any school experimentation come from?
- What constituted any barriers and/or obstacles?
- In what ways has the teaching of mathematics changed over the last five years?
- In what ways has available technology been integrated into the math classroom?

Content Questions:

- What mathematics courses are available to incoming students?
- What support systems are available for students who have difficulty?
- How is the school moving toward an integrated sequence of mathematics courses?
- Have you noted any increase in the number of students taking college preparatory mathematics beyond the first year?
- What provisions are made for special education and limited English proficient students?
- What do you see as the next step to be taken in terms of math reform at this school?

Pedagogy Questions:

- If we were to walk into a math classroom right now, what kinds of things would we be likely to see?
- How aware are your teachers of recent findings about how students learn? How is this knowledge influencing their use of new pedagogical techniques such as cooperative instruction, problem solving, manipulatives, and the infusion of technology?
- How are students assessed in their math classroom?

Attitude Questions:

- How would you rate your teachers' understanding of the Equity 2000 goals?
- How would you rate their commitment to those same goals?
- Which of those goals, if any, do your teachers feel most strongly about?
- What changes over the last five years have you seen in the teachers' expectations for their students?
- What do you consider the teacher's role to be in achieving and sustaining equity?
- What role do you feel this project has played in bringing about the changes in content, methodology and attitudes that you have described to me?
- What are the needs now, and how could such a project as this help to satisfy them?

Results

The responses to the interview questions were examined for patterns; a selection of the results will be discussed here in the categories outlined in the questions above, with one exception. When results were analyzed, it made more sense to group responses in the areas of content, pedagogy and attitudes, and integrate responses to general questions into these three categories.

Content. All of the schools managed by the principals interviewed had all of their 9th graders in an algebra program. None of the schools was teaching any kind of pre-algebra, Math A, or any other kind of math skill-building course. Algebra was the only selection for incoming students. One of the schools had implemented this policy and program for the 1995-1996 academic year, three were in their third year, one was in its fifth year and one was in its 6th year of implementation. In view of the curriculum materials being used, all of these schools are moving toward use of more innovative curriculum materials which attempt to integrate the various strands of mathematics as called for in the NCTM Standards (National Council of Teachers of Mathematics, 1989).

As all students are placed in a minimum level course of algebra 1/course 1, it is inevitable that some students may experience difficulty. The principals interviewed reported the use of some kind of tutoring program to ensure student success. These efforts combined use of teachers, students, college students and parents. Each of the six schools had an after-school program, although one had a pull-out program during the day as well. One school had a special education class using new curriculum materials, and three schools used new materials in sheltered classes [those designed for limited English proficient students].

Although the principals did not report a better success rate with the implementation of the "algebra for all" policy, they did recognize that the success rate in algebra 1 was no less than it had been in remedial courses; therefore many more of those students succeeding were now doing so in course work of more depth and rigor than they had previously. In fact, in one district, more African American and Latino/Latina students were passing algebra 1 than had ever taken algebra 1 four years ago.

Five of the schools had had "algebra for all" for more than one year, so they were able to reflect on the numbers of students continuing on to higher levels of mathematics. All of these schools reported increased numbers of students going on to higher level mathematics courses, one of the main goals of Equity 2000. The principals felt that the new curriculum materials, stressing problem solving and applications of mathematics to the real world, helped their students see the relevance of mathematics and encouraged that continuation onto higher level courses.

Pedagogy. The principals' responses to questions on pedagogy reflect either a reluctance to be very specific or a lack of knowledge of what actually is going on in the mathematics classrooms in their schools. Or perhaps our questions were not designed well enough to elicit the kind of information we sought.

There were some reflections on pedagogical aspects of the new curriculum materials being used. One principal felt that cooperative learning was important as it would be important to how most young people would function in their future employment. Two principals saw benefits to the heterogeneous groupings resulting from the elimination of tracking, reporting fewer gang-related problems and higher self-esteem among minority students since implementation took place. Another principal noted that she felt that oral and other alternative assessment techniques were often better for minority students (especially Hispanic), both in terms of their immediacy and their suitability to the culture and background of these students.

However, there were some concerns about these same instructional practices. While two principals felt that alternative assessment techniques were being used well by their math teachers, two others reported their teachers as slow to get started with many of these techniques. And although only one principal felt that more support for cooperative learning strategies was important, we judged that, in the other schools, the principals lacked knowledge of the depth of understanding of a set of skills needed by teachers in order to fully empower their students through cooperative experiences.

Another small indication of a lack of knowledge of the mathematics reform called for in the NCTM Standards (NCTM, 1989) was the fact that one of the six principals felt that the technology on site was adequate for instructional needs; s/he was referring to Apple IIE labs.

Attitudes. Because of the different implementation timelines, the principals saw the role of the project in two different ways. Those whose schools were in their fifth or sixth year of implementation saw the role of "Building Bridges" as the support for their school's own decision to implement the new curriculum programs, while those whose programs had been implemented more recently felt the project had provided the leverage for reform. The one belief common to all six of the principals was that Equity 2000 was the critical piece in sustaining the reform effort. None felt their school would have an "algebra for all" policy today without the staff development and support rendered by the project.

All of the principals felt that 100% of their math teachers had a good understanding of Equity 2000's goals and of their importance, a finding corroborated by Becker, Pence and Pors (1995). These goals, however, were supported by a low of 70% of the math staff at one site to a high of 90% of the staff at another. Even if not fully committed to Equity 2000, the principals strongly felt that the project had been essential in getting math faculty to work together better than they had before entering the staff development program. They all saw a more common focus shared by mathematics staff, as well as more time spent by the staff in discussion of how to go about best meeting student needs. One principal said that her staff was also more open to meeting with math teachers at other high schools as well as with those teaching at other levels. More educational conversation and more cohesiveness were highlighted by principals as ways in which their mathematics faculties had evolved as a result of their project involvement.

Each one of the principals described their math faculties as split 80%-20% in their commitment to the NCTM Standards (NCTM, 1989) and the implementation of their own schools' new mathematics programs. In one school, two teachers were not using new materials. In the other five schools, the uncommitted 20% were working at the use of new curriculum but would bale out if given the opportunity. Principals felt that these teachers had gone along because of peer pressure and strong support, both provided through Equity 2000 strategies. Such project support, the principals felt, would continue to be instrumental in keeping such instructors on the road to reform.

In addition to teacher attitudes, the principals noted a better attitude toward math on the part of students. Although not aware of gender equity as an issue, which again might reflect a lack of knowledge, they did view implementation of the new math programs as having had a major effect on equity for students traditionally underrepresented in mathematics.

Next Steps. There were a number of areas of concern expressed by the principals as they reflected on next steps needed to continue the move toward reform in mathematics. While the "Building Bridges" program had put particular stress on implementation of algebra 1/course 1, the principals saw a real need to provide continued staff development to support changes in higher level courses, especially the broadening of instructional strategies and the integration of technology. Another principal thought there was a need for closer articulation with feeder schools, especially with respect to the coordination of mathematics materials. Interdisciplinary strategies were raised by two other principals as the next step in making mathematics connections across the curriculum.

Some "next steps" that the principals may not be aware of the need for became apparent in the interviews. Of the four schools using the College Preparatory Mathematics program, only one principal recognized that this program is transitional and that schools using it need to be planning evolution to other, more innovative materials. Secondly, the principals were very vague in responses to questions dealing with the success of the programs in their current state of implementation. It would seem that some evidence that their programs are going in the right direction is essential.

Conclusions

The observations of these principals give us a glimpse of what a project supporting mathematics reform might need to include. It is clear they felt that such an effort must continue to support teacher risk-takers, concentrate on continued training in instructional strategies and the infusion of technology, and help practitioners structure an environment that seeks to include every learner and value diversity. Further, some of the principals' own responses may point to a need to better inform them, so that they too may seek to facilitate reform efforts.

One of the most strongly held beliefs about the value of this project, in the minds of all six principals interviewed, was that it got their math faculties talking with each other about the education of their students. This "educational conversation" produced, they felt, more cohesive departments, a more common focus, more shared values and a more effective approach to mathematics education. It may now be time to *extend the educational conversation* to others who, in turn, can help to expand, support, sustain and refine the mathematics reform effort. In the opinions of the principals interviewed, among already proven values of Equity 2000 has been its ability to engender just such conversations.

References

Becker, J.R., Pence, B.J. & Pors, D. (1995). Building bridges to mathematics for all: A small scale evaluation study. In Owens, D.T., Reed, M.K, & Millsaps, G.M. (Eds.). Proceedings of the Seventeenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, vol. II. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 255-61.

Cooney, T.J. (1993). On the notion of authority applied to teacher education. In Becker, J.R. & Pence, B.J. (Eds.) Proceedings of the Fifteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, vol.I. San José, CA: The Center for Mathematics and Computer Science Education, 40-46.

Cooney, T. & Jones, D. (1988). The relevance of teachers' beliefs for research in mathematics teacher education. Paper presented at the Sixth International Congress on Mathematical Education, Budapest, Hungary.

Ernest, P. (1991). The impact of beliefs on the teaching of mathematics. In Dossey, J.A., Dossey, A.E. & Parmantie, M. (Eds.). Preservice teacher education: The papers of action group 6 from ICME 6, Budapest, Hungary. Normal, IL: Illinois State University, 237-242.

Grouws, D. (May, 1988). Overview of the role of research on mathematics teaching and learning in improving research in classroom instruction and teacher education. Paper presented at the First Wisconsin Symposium on Teaching and Learning Mathematics, Madison, Wisconsin.

National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: Author.

Peluso, T., Pence, B. & Becker, J.R. (1994). Developing a community of risk takers. In da Ponta, J.P. & Matos, J.F. (Eds.). Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, vol. II. Lisbon: University of Lisbon, 56-63.

Romberg, T. & Carpenter, T. (1986). Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In Wittrock, M. (Ed.). Handbook of research on teaching. NY: Macmillan Publishing Col.

The Teaching of Mathematics from within the School. Teachers and Principals as Researchers

Patricia Perry, **Pedro Gómez**, **Paola Valero**

"una empresa docente" - Universidad de los Andes, Colombia[*]

This paper reports on an action-research study with ten state schools in Bogotá, Colombia. Its purpose was to explore the issues involved in secondary mathematics education from an institutional point of view. The information collected from the interaction with the principal, the mathematics headmaster and two teachers from each school who participated in a professional development strategy was analyzed with a hermeneutical research methodology in order to construct (on the basis of the systemic approach) a model of the Institutional System of Mathematics Education, and to determine the initial and final states of the system. Relevant differences were found between the two states. However, it remains to be seen whether these changes are permanent.

Institutional approach to mathematics education

Some recent studies show a trend towards exploring the issues concerning the teaching and learning of mathematics, which are different from those closely related to the interaction among teachers, students and mathematics knowledge within the classroom. In the US, for example, several projects have undertaken the issue of improving mathematics education within schools in a context of educational reform. The Urban Mathematics Collaborative (Webb & Romberg, 1994; Heck, 1995) seeks to generate a change in urban mathematics teachers by stimulating teamwork among teachers of different schools. The School Restructuring Study (Secada et al., 1995) tackles the issue of how to reform mathematics at the secondary level, in educational institutions and mathematics departments, with the aim of developing balance and productivity among efforts in these two environments. Other projects like The Coalition for Essential Schools, the New Standard Project, Goals 2000 and the NSF State Systemic Reform Initiative have taken this approach, most of them from the point of view of systemic reform (Cohen, 1995). On the other hand, The Teachers As Researchers Movement, closely related to the above issues, has gained a place in PME through the creation of a working group (Mousley, et al., 1995).

In Colombia, the 1994 General Education Law and, with it, the whole plan of curricular decentralization, have opened a large space for educational institutions to assume leadership in the endeavor of improving quality in mathematics education. Within this context, this study approached some of the issues involved in mathematics education from an institutional point of view. This institutional approach stressed the importance of a coherent practice from principals, headmasters and teachers in the

---

[*] The research reported in this paper was supported by the Corona Foundation and the Colombian National Ministry of Education.

construction of a process of change that could influence the quality of mathematics teaching in schools. As a means to face the issues of school mathematics, analyze them and influence them, a professional development strategy involving principals, headmasters and teachers of the participant schools, was implemented. It was based on the conceptual and methodological principles of action–research.

This professional development strategy was based on three assumptions. Firstly, the school is the space where the relationships among the agents closely related to mathematics education issues (heads and teachers) are built and evidenced. Coherence between institutional plans and instructional practice might lead to an institutional environment that can favor the quality of mathematics teaching and learning. Secondly, change is made by people and this requires a re–structuring process of their belief systems and of their administrative and pedagogical knowledge. Finally, action–research, as a way of encouraging critical reflection and enhancing the agents' ability to detect and propose solutions to problems that depend on them, is seen as an appropriate method to initiate such a change.

In what follows, the professional development strategy is briefly described, the methodology used is outlines, a model of the Institutional System of Mathematics Education (ISME) is presented and, based on that model, the initial (before the strategy) and the final (after the strategy) states of the system in the participant schools are described.

Professional development strategy

Ten state schools participated in the study. Two teams were formed with participants from each school. The first one included the principal and the foreman of the mathematics department. The second one included two secondary mathematics teachers.

Both, heads and teachers in their own teams, went through the experience of carrying out a process of action-research. Heads had to identify a particular aspect of the issues related to mathematics in their own schools, which they had the power and the will to modify. Concerning this aspect, they planned a specific action aimed at a change. Then, they implemented it, observed its results and evaluated the effects it had upon the aspect under consideration. Teachers —either individually or in pairs— had to choose a topic from the syllabi of their courses at that time, upon whose teaching they wanted to improve. It should span, at the most, three class sessions. They completed the corresponding curricular design and development.

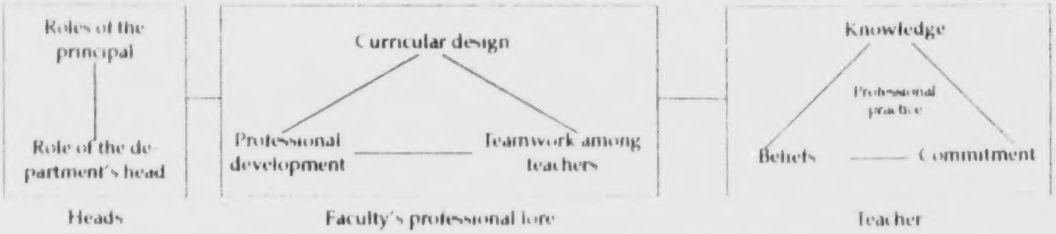
After working for eight months, the teams assembled for presentation of results and writing of report papers on their experience. Several general meetings were held for both teams during the eight months. During these meetings heads and teachers, in their corresponding groups, were given some conceptual and methodological tools for their project, and had the opportunity of sharing and discussing the different stages of their project with their pairs. Furthermore, several individual meetings were held between heads and teachers from each school with the researchers developing the study in order to discuss problems or doubts concerning their projects.

Methodology

Information was collected in the form of field notes, tape recordings, interviews and documents written by the participants during the professional development strategy. This information was used to construct the ISME model and describe the system's states. An hermeneutical research methodology (Addison, 1992) was used. The three researchers involved themselves in an iterative scheme of generation of transient versions of the model and the states. Each new version was considered under a critical dialogue, and nourished by the experience lived in the interactions with the participants in the professional development strategy. Every opportunity for interaction was taken as a new chance for generating a better and broader understanding of the reality under study. Thus, at a certain stage of the process, a certain proposal was tested on the basis of the researcher's maturing readings of reality. Critical discussion of the degree of adequacy of the proposals to the reality under examination engendered new ideas about its fundamental features. These ideas, deliberation about their particular insights and an attitude of pursuit of an objective consensus within a milieu of critical dialogue, allowed the researchers to devise a new version of the model and its states. At this point a new iteration of the whole process started. This methodology bears some resemblance to the competitive argumentation method proposed by Schoenfeld and colleagues (1993) with which several researchers, with different interpretations and departing from sometimes contradictory evidence, involve themselves in a process of debate in order to produce coherent consensual explanation of such evidence.

Institutional System of Mathematics Education

A systemic approach was used in order to understand and simplify the complexity of the institutional reality related to mathematics teaching and learning and as a means to reveal the acting elements, their meaning and the very gist of their actions and the potential effects of changing a given relationship among them (Artigue, 1988). Thus, the idea of the Institutional System of Mathematics Education (ISME) was introduced to represent this complex reality. Following the methodological principles described above, a model of the ISME was constructed corresponding to a conception of the relevant aspects of that reality (see figure). It is acknowledged that this is one among many other possible models of the same reality.



Institutional System of Mathematics Education Model

In what follows, each element of the model is briefly described (See Valero et al., in press, for a more detailed description of the model and of the system ideal state).

Regarding heads (principal and mathematics foreman) two main elements may be shaped. The *roles of the principal* refer to the principal's leadership and supporter roles corresponding to his/her understanding of the organization structure, workings and planning, and his/her ability to supply the necessary resources, organize and commit people into relevant projects (Furtwengler, W. & Hurst, D., 1992). The *role of the of the department's foreman* refers to the way he/she assumes a leadership role amidst the mathematics faculty. The mathematics faculty's professional lore refers to the customs, ways of life, qualities, trends and preparation concerning the teaching of mathematics inasmuch as shared by the institution's mathematics faculty (Rico, 1990, pp. 36-40; Hyde et al., 1994, pp. 49-50). It reveals itself through three main aspects. The *curricular design* as the previous definition of an "operational plan for instruction that details what mathematics students need to know, how students are to achieve the identified curricular goals, what teachers are to do to help students develop their mathematics knowledge, and the context in which learning and teaching occur." (NCTM, 1989, p. 1). *Professional development* referring to the institutional opportunities for teachers to learn and increase their specialized knowledge of mathematics as well as of its teaching (Rosenholtz, 1989). *Teamwork among teachers* as the teachers' willingness and attitude towards exchanging assistance with their colleagues in questions related to mathematics teaching (Rosenholtz, 1989). Concerning the mathematics teacher as an individual, the model takes into account three main aspects: The teacher's *beliefs* about mathematics, their teaching and learning, his *knowledge*, not only about mathematics, but also about mathematics education as his professional discipline, and his *commitment* to his practice, in terms of how much he gets involved, cares and actually carries out. Finally, all the elements mentioned above influence the teacher's professional practice. Nonetheless, this factor is not considered as an element of the system because it was not analyzed, nor directly influenced during the professional development strategy.

Initial and final states of the ISME

Based on the above model, the information collected in the interaction with the participants, and the methodological approach previously described, the initial and final states of the system were determined. These states designate the values shown by the elements of the model before and immediately after the implementation of the professional development strategy.

Principal's Roles

Initial State. Principals had difficulties in designing projects and leading their execution. They partook of a firmly-established tradition of doing things by coercion even if they made no real sense for their particular institutions. They had very vague ideas about the issues related to mathematics teaching and learning in their schools. They did not have a clear insight of the incidence of institutional aspects upon the problems and could not face up their responsibilities in this concern. They were not mindful enough of the importance of the academic functions of the mathematics foreman. Their relationship whith their own mathematics department foremen was centered upon administrative issues more than upon academic ones. They supported professional development through strategies such as encouraging attendance at conferences and traditional in-service training courses, but were far from believing that the faculty themselves could carry out their own professional development as a part of their scholastic activities, with the institution's support.

Final State. Most of the principals concluded their action–research projects. In some cases, principals broadened their knowledge about a particular aspect of the issues affecting mathematics in their own schools and faced up to their accordant responsibilities in them. In general, principals became more perceptive about the meaning and purpose of their faculty's professional development and encouraged and induced their teachers to work collectively. In some cases, principals made contributions to the improvement of the department meetings and showed concern, on behalf of the institution, about their faculty's professional practice and about the codes and usage that guide it. They helped improving their teachers' commitment to teaching.

Department's Foremen Role

Initial State. Foremen lacked awareness about the leadership they could and should be assuming within their faculty. Their relationships with the principals were confined to administrative issues. They did not influence upon the faculty's professional development or teamwork. Their duties were limited to organization of mathematics curricular design and implementation. They had little impact upon their colleagues' commitment to the accomplishment of adopted plans.

Final State. Foremen did not show signs of developing an academic relationship with their principals. They kept adopting passive attitudes, lacking leadership as well as any status of spokesmen of their teachers in front of the principals. They still were not inducing their faculty to teamwork. Nevertheless, they acknowledged among their main responsibilities the devising of an institutional proposal for mathematics schooling. At some schools, foremen were starting to bear well upon some teachers' commitment to their practice.

Faculty's Professional Lore

Initial State. Heads, teachers and, in general, schools participating in the study lacked awareness of being immerse in an institutional lore. Its development and upgrading were not born in mind when decisions about plans and their implementation were being made. Teachers did not consider themselves as members of a national community of professionals in mathematics education. Overall, institutions lacked operational plans concerning the field of mathematics education. Their faculties lacked knowledge, clarity and consensus about the goals to be achieved, the methodologies to be used and the evaluation procedures. Teachers, as well as heads, were adapted to faculty qualification strategies whose main feature is the passive transmission and reception of information. Professional development was managed through traditional in-service training courses, and was envisioned within the institution (namely, as a personal enterprise, outside the interests of the institution, requiring interaction with somebody alien to the faculty), without necessarily serving mathematics curricular design and development. Teamwork among teachers was roughly absent. Faculty free-hours were scheduled so that they conflicted with any opportunities for teachers to exchange assistance among them.

Final State. Mathematics faculties at most of the participant schools began cooperating, assuming teamwork tasks, holding discussions, and showing interest in familiarizing themselves with mathematics education topics. This movement even reached teachers that did not participate in the professional development strategy. On the other hand, by the end of the study all participant teachers (many of them for the first time) have had an opportunity to write progress reports and final reports of their own projects. In addition, several of them had written short papers where they presented their small action-research work. Many of the participants had the occasion to witness, for the first time, the actual existence of mathematics education as a young discipline on the move that wields concepts, theories and techniques of great assistance in tackling the difficulties that emerge in the classroom. All participants realized the significance of attending qualification programs that contribute meaningfully to curriculum design and development at their institution. Teachers noticed that different qualification plans influence diversely on their teaching competence. Institutional lore concerning mathematics schooling had begun awakening due to the new experiences with teamwork. The latter had also increased the teachers' degree of commitment to their professional practice. In turn, the commitment stimulated by the professional development strategy led teachers to teamwork.

Mathematics Teacher

Initial State. Teachers were not aware of holding any view about mathematics and its schooling. They also lacked any apparent pictures of alternative views concerning these topics. No survey was carried out of the teachers' knowledge of mathematics. Nevertheless, interaction with them revealed serious deficiencies in mathematics teaching know-how and in the knowledge of the field of mathematics education. The commitment teachers showed to the activities offered during the execution of the professional development strategy was illustrative of their commitment to professional practice. Teachers willingly associated with the study, even when forewarned that they could not expect any financial or academic bonuses from such an association. Looking for solutions to their problems was their major source of motivation and commitment.

Final State. Teachers had the opportunity to realize they do hold views about mathematics and its schooling, just as all other mathematics teachers do. From this recognition, they generated a self–questioning process.Through the activities conducted during the study, teachers had a chance to come upon (what for them represented new) information about mathematics, mathematics teaching know-how and the field of mathematics education. Although they found themselves running into a program that was far from satisfying their first expectations, most of the teachers worked intensely, taking even pains that were not required of them, and attended the bulk of the meetings. They gained awareness of their responsibilities and their commitment to professional practice. They were left in a state of uncertainty concerning how to assume such commitment onwards.

Conclusion

The first result of this study is the proposal of the model for the Institutional System of Mathematics Education. This model is seen as a simplification of a complex reality through the selection of the structural characteristics of the system that were found to be relevant to the eventual improvement of teachers' practice. It presents only the direct structural relationships among elements. It conforms to a particular ideological position towards what the system should be, and, in that sense, it is only one of many possible models of the reality in question.

The professional development strategy departed from traditional in–service teacher training courses. On the one hand, it involved the principal and the mathematics headmaster. On the other hand, it was centered on the design, development and evaluation of action–research projects produced by each participant.

The results show that this strategy had preliminary effects on the prinicipal's role, the faculty's professional lore and the teachers' commitment. Whether these changes are permanent and whether the final state is a stable state of the system, remains to be seen. Similarly, it is too early to know whether the changes observed may have had any effects on the other elements of the system, and, in particular, on teachers' practice in the classroom. Nevertheless, a follow-up made six months after the end of the study has shown that, in most schools several changes are still on effect, and new ones have taken place. Firstly, principals continue taking an active leadership role in academic issues concerning mathematics teaching and learning in their schools. Secondly, the action–research strategy to professional development has been transferred to other knowledge areas in some schools. Thirdly, in most schools, the mathematics department, as a coherent and dynamic group, has produced written proposals for new action–research projects in mathematics education. This reveals a continuing interest and commitment from mathematics teachers, foremen, and principals on the development of the faculty's professional lore and on the improvement of the teaching practice.

References

Addison, R. D. (1992). Grounded hermeneutical research. In Crabtree, B.F., Miller, W.L. (Eds.) *Doing qualitative research*. Thousand Oaks: Sage, pp. 110-124.

Artigue, M. (1988). Ingénierie didactique. *Recherches en Didactique des Mathématiques*. 9 (3), pp. 281-308.

Cohen, D.K. (1995). What is the system in systemic reform? *Educational Researcher*. 24 (9), pp. 11-17.

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: a model. *Journal of Education for Teaching*. 15 (1), pp. 13-33.

Furtwengler, W.J., Hurst, D. (1992). Leadership for School Quality–Personal Challenge, the Missing Factor. *Paper presented at the Annual Meeting of the American Educational Research Association*. San Francisco.

Heck, W.P. (1995). Communities of teachers in urban mathematics collaboratives. Guest editorial. *The Mathematics Educator*. 6 (2).

Hyde, A., Ormiston, M., Hyde, P. (1994). Building professional development into the culture of schools. In Aichele, D.B., Coxford, A.F. (Eds.) *Professional development for teachers of mathematics*. Reston: NCTM, pp. 49-54.

Mousley, J., Breen, C., Zack, V. (1995). The teachers as researchers in mathematics education. In Meira, L., Carraher, D. (Eds.) *Proceedings of the 19th PME Conference*. Recife: Universidade Federal de Pernambuco, pp. I-173.

NCTM (1989). *Curriculum and evaluation standards*. Reston: NCTM.

Rico, L. (1990). Diseño curricular en educación matemática: una perspectiva cultural. In Llinares, S., Sánchez, M.V. (Eds.) *Teoría y práctica en educación matemática*. Madrid: Alfar, pp. 17-62.

Rosenholtz, S.J. (1989). *Teacher's workplace: The social organization of schools*. New York: Longman.

Schoenfeld, A.H., Smith, J., Arcavi, A. (1993). Learning: The microgenetic analysis of one student's understanding of a complex subject matter domain. In Glaser, R. (Ed.) *Advances in instructional psychology*. Hillsdale: LEA, pp. 55-175.

Secada, W. et al. (1995). The royal road to secondary mathematics reform: Department initiative or school restructuring. *Paper presented at the Annual Meeting of the American Educational Research Association*. San Francisco.

Valero, P., Gómez, P., Perry, P. (In press). La problemática de las matemáticas escolares desde una perspectiva institucional. In Gómez, P., Perry, P. (Eds.) *La problemática de las matemáticas escolares. Un reto para directivos y profesores*. México: una empresa docente y Grupo Editorial Iberoamérica, pp. 3-51.

Webb, N.L., Romberg, T.A. (1994). *Reforming mathematics education in America's cities. The urban mathematics collaborative project*. New York: Teachers College.

TEACHERS' CONCEPTIONS ABOUT MATHEMATICAL ASSESSMENT

George Philippou & Constantinos Christou
Department of Education
University of Cyprus, 1678 Nicosia, P.O. Box 537

Abstract. The purpose of this study, which is part of a larger project, was to survey teachers' conceptions about assessment: its role, functions and techniques practised in the mathematics classroom. Responses by a substantial proportion of fifth and sixth grade mathematics teachers to a mailed questionnaire revealed a rather optimistic picture in most statements of a specially developed scale. Although this particular group of teachers seems to be well aware of contemporary views about assessment, a few interviews shed more light and uncovered significant misconceptions and different understandings and interpretations of basic concepts and definitions. It seems that most teachers have a rather confused idea about some basic concepts and terms used in the survey.

Assessment is no longer considered as the final act of the teaching/learning process; it is rather viewed as an integral part of instruction (NCTM, 1989) and a necessary means for designing and guiding appropriate class activities and interactions. The dominance of standardised tests, evident until recently in some of the leading countries of the world (Stiggins & Bridgeford, 1985), is now giving way to teacher developed tests and alternative non-test assessment methods. As the wave of reform sweeps across many countries, it becomes evident that improved mathematics teaching ultimately relies on the quality of teacher-student interactions. The teacher is by far the key factor in any effort to improve classroom activities, while the role of assessment in ensuring that these interactions are meaningful learning situations is crucial (Webb, 1993).

The growing interest in mathematical assessment and the recently accumulated research results have led to clearer and functional definitions of related concepts and to the development of new authentic assessment techniques (NCTM, 1989). If, however, the bright new ideas are to find their way into the actual mathematics classrooms, it is imperative that we secure and carefully consider a deeper understanding of classroom teachers' views, beliefs, conceptions and practices so that we can develop appropriate programmes to change the existing traditional teaching.

From the existing body of research, it has been rather well established that teachers devote a considerable proportion of their time to assessment related activities (Stiggins, 1988), and that assessment techniques vary by subject and age level. There is also some evidence (from Greece) that teachers have rather positive attitudes toward assessment (Bamboucas & Troullis, 1993). Yet, despite the volume of related research, it seems that there is still limited understanding of the ways in which teachers practise and evaluate their assessment methods, the manner in which they view the role and the relation of this process to the total mathematics instruction, and the nature of their beliefs and attitudes. Even though the emphasis placed by researchers and educators on assessment is hardly second to any other aspect of instruction, the affective dimension of the process has not been studied enough, as it has been, for instance, in the case of teachers' conceptions and attitudes towards "problem solving".

The purpose of this study was to investigate further practising teachers' conceptions about this aspect of the teaching-learning process, in order to broaden our understanding of the classroom "ecology" and hence make the effort of restructuring mathematics instruction more meaningful. In particular, the following research questions were formulated:

1. How do teachers view the role of assessment?
2. What are the criteria most frequently used by teachers when grading students?
3. What is the item format most commonly used by teachers in their tests?
4. To what extent does assessment support or relate to instructional objectives?
5. Which are the types of items most commonly used by teacher in their testing?

Methodology

The study was designed to include two levels of investigation: First, the mean value or quantitative method of collecting and analysing data from a representative sample of the target population was used. Second, in order to penetrate deeper into teachers' thoughts, beliefs and conceptions, a small number (8) of semistructured clinical interviews of selected teachers were transcribed and analysed.

The Instrument developed by the researchers for the first part of the study, consisted of a four section Likert-type scale with statements allowing for four possible alternatives (1 for no emphasis placed and 4 for major emphasis), and a fifth part requesting the subjects to specify by type the items commonly used in their self-developed tests. Each of the first four sections corresponded to one aspect of assessment as specified above i.e. i) the **Role** or function of assessment (5 items), ii) the **Criteria for grading** students (6 items), iii) the **Item format** used in teacher prepared tests (12 items), and vi) the definition and emphasis of **Instructional objectives** (to asses the integration between assessment and instruction) (7 items). The focus of the fifth section was a description of the **Type of items** used in class assessment, thus indicating the level of consistency of professed beliefs with actual practice in assessment.

The questionnaire was mailed to the total population of fifth and sixth grade teachers in Cyprus-the reason for this selection being the fact that those are currently the only grades of elementary school in Cyprus where marks are assigned. A percentage of 86% (610) of this population (more than satisfactory) responded by returning completed questionnaires, of which 68% were females and 32% males, 57% were teachers, and 43% were higher rank teachers-headmasters and deputy headmasters. The proportion of higher ranks seems bigger than the normal lower/higher rank ratio, but this is quite natural for the target population, i.e. the school teachers of the upper two classes.

The responses were classified as positive when the subject's preference was on the right half of the (linear) scale, i.e. when 3 or 4 was chosen. The "consensus criteria" defined by Pehkonen (1993) were used, according to which "complete consensus" means that a proportion greater than 95% of the responses is on the positive side of the continuum, "consensus", means that the proportion is between 85% and 95%, "near consensus" that the proportion is between 75% and 85%, and "lack of consensus" that the proportion falls below 75%.

Results

Responses of the subjects to the major part of the items of the four scales are given in the Appendix. The percentage in the final column refers to the agreement responses i.e. the sum of columns 3 and 4. In this section an analysis of responses by scale section together with comments and extracts from interviews is included.

An awareness of the Role of assessment was the first aspect of interest. The items of the relevant section of the scale measured responses to the question "what is the role of assessment in modifying instruction, developing the curriculum, diagnosing students difficulties, assessing teaching- learning process, and grading the students?" (Space was also provided for subjects to include any other aspect they wished). The subjects expressed "complete consensus" on two out of the five items, one being that the role of assessment is "to find out students' learning difficulties" and the other" to assess the effectiveness of instruction" (by almost unanimous agreement, 99% and 96% respectively). On the contrary, they did not agree (lack of consensus) that assessment functions include: "modifying instruction", "assigning grades to students", and "modifying the curriculum" (the agreement proportions were 73%, 53%, and 46% respectively). It seems that the highly centralised educational system provides a serious constraint not allowing teachers to take initiatives concerning the curriculum content, even though they can see the necessity. During interviews all teachers mentioned the diagnostic role of assessment and particularly "providing appropriate guidance to students-therapeutic work", but they were reluctant to point out the relevance of assessment to planning of subsequent lessons. Concerning their eligibility to change the curriculum and in essence the subject matter they felt powerless. To use the actual words of an experienced teacher:

- A. Because the subject matter is not in my authority.
- Q. Is it prescribed from above?
- A. Absolutely, is not for us. I wished I could.
- Q. Really? Are you not allowed, for instance, to leave some parts behind?

- A. Well ... certainly. You see, nobody can follow exactly what a teacher does in the closed class, but ...

Criteria for grading. Grading the students is frequently conceived as equivalent to, confused with, or at least thought of as the major function of assessment and evaluation. The subjects in the study were required to indicate the extent to which each of the following six criteria is used in their grading: "class participation", "performance in classwork", "test scores", "homework assignments", "student's effort", and "persistence". Their responses indicate that they use a variety of means or criteria. Indeed the "complete consensus" level was observed in five of the section items (homework assignments was excluded). Teachers expressed their distrust to homework as a reliable mean to grade students. However, a degree of uncertainty came to the fore during interviews, as the major characteristic of participating teachers, concerning assigning grades to students. The following is an extract from the interview with a competent female headtmaster:

- Q. Which factors do you primarily consider in determining students grades?
- A. His/her ability and understanding.
- Q. Could you be specific? How about written tests?
- A. Yes, ... as one factor.

A less experienced teacher puts emphasis on the daily interaction with students, he answered the question as follows: “well, I form a picture for each student from the daily contact with them, how clever their participation is. If I know that somebody worth an "A" , I' m not going to change my view even if he/she fails in 2-3 tests”.

The composition of teacher-developed test, with a variety of Item format, is indicative of the depth of their awareness about recent developments on assessment techniques and the degree to which they teach for higher order thinking and (non-routine) mathematical problem solving. The relevant scale included 12 items and the results are summarised as follows:

- The subjects did not seem to value the so called objective items, "true-false", "matching", or "multiple-choice", the agreement proportions were 33%, 40%, and 55% respectively. They also placed limited emphasis on items requiring "knowledge of concepts and definitions" and items "requiring understanding", the positive responses found to be only 38% and 56% respectively. Lack of consensus was also observed on "problems similar to textbook", this being a positive reaction, but the same phenomenon was also observed on items requiring "explanation of the solution strategy" and "novel problems".
- Near consensus response was observed on items requiring "application of procedures", involving only "minor variations of textbook problems", and on "problems with more than one answers". This means that teachers indicate a positive tendency towards de-emphasising operations and procedures, and an

attempt to include in their tests problems not similar to the textbook, as well as items with more than one answer.

- Consensus was found on items requiring "application of concepts to novel situations", a characteristic which is in line with contemporary teaching objectives.
- Complete consensus was expressed on none of the items of this section.

At least one contradiction is obvious at the first glance, since any "application to a novel situation", is definitely a "novel problem", hence the gap of two levels between these items (non consensus and consensus) can not be justified. At Clarification of this inconsistency was one of the objectives at the interviews. Most participants were able to state the defining characteristic of a novel for the students problem, but only two out of eight were able to give a good enough example. A similar finding applies also for the meaning of the phrase "knowledge of concepts" and "problems with more than one answers". The latter was confused with "open problem".

The items testing the relationship of assessment to Instructional objectives called upon subjects to report the emphasis they placed on assessment related instructional objectives. The responses indicated complete consensus on the following four statements: "concept understanding", "developing of investigative strategies", " problem solving" and "applications to everyday problems". The consensus level found on the remaining three objectives measuring emphasis on "oral and written communication", "developing positive attitudes" and "quick and correct application of procedures" is also considered positive. However, when the interviewees were required to explain how they develop "oral and written communication" they all complained that written reasoning is time consuming.

Q. Do you expect students to explain in words what they do?

A. Yes, quite regularly during class activities.

Q. I meant written, when you are testing your students.

A. No, they don't have enough time. Besides, it will not be easy for most of them. See, they can not express their thoughts well ...

Classroom assessment practice was surveyed by recording the emphasis placed upon different item types included in teacher-developed tests. Responses were considered as positive if at least 40% of the test items were stated to involve the specified characteristic. Three out of the seven item types included in the scale could be classified as "traditional", in the sense that they placed emphasis on traditional teaching objectives. The three traditional items involved emphasis on "procedures and algorithms", "similar to textbook tasks" and "single answer questions", and the proportions of agreement were 29%, 36% and 46% respectively. The non-traditional items involved emphasis on "concept understanding", "requiring justification of solutions", "application to novel situations", and "requiring explanation of solution plan". The proportion of positive responses were 32%, 42%, 45%, and 58% respectively.

Although a difference between the traditional and non traditional items mentioned above is evident, the results on this scale tend to run against the rather optimistic conclusions drawn from earlier sections. When it comes to reality it appears that the professed acceptance of contemporary ideas remains empty words. It is striking that although teachers unanimously (98%) state that in their instruction they place major emphasis on concept understanding (to mention but one inconsistency), at the same time 68% of them admit using less than 40% of items testing the realisation of the same objective. This argument is strengthened further by the findings of the analysis of the interviews. In fact, many parts of apparently positive findings tend to become doubtful by the findings from the interviews of teachers. For instance one of the teachers claimed that she includes only a minimum number of items similar to textbook, yet she stated that her students are taken by surprise. The following is part of the conversation:

Q. How do you manage to have so many non similar to textbook test items?
A. Yes, it's time consuming, but this is the objective, to have some variation.
Q. And what is the student's reaction, when they phase them in a test?
A. You mean if they complain. The truth is that they do, they say that we never met such a problem before.
Q. So, you come to accept that you are asking for too much?
A. They will finally get used to that.

To test for possible differences in the scale responses by sex and rank order the multivariate analysis method was applied to data. The comparison showed statistically significant differences in favour of the female subjects in four items of the **Criteria for grading** scale and in one item of the **Item type** scale. Similarly, higher rank teachers were found to hold significantly better conceptions in three items of the **Role of assessment** scale, in four items of the **Item format** scale, and in three of the **Instructional objectives** scale. It is worth noting that female teachers put significantly more emphasis on the development of positive attitudes than the male teachers. No statistically significant interactions were noticed between sex and rank position.

Discussion

Much of the responsibility for implementing change "falls squarely on the shoulders of classroom teachers" (Cain et al., 1994). Everyday teachers gather information about their students' mathematical knowledge, which they use some way or other in making significant instructional decisions concerning the design of learning activities. In this study, we have tried to draw a map of some of teachers assessment conceptions and practices, limiting the scope of our investigation only to formal evaluation methods.

Some of the findings seem to support the claim that the subjects are quite aware of and accept contemporary ideas about the role of assessment, the criteria for grading students, the format of the items to some extent, and the instructional objectives. It should, however, be noted that although this awareness is useful and required, it does not suffice. Knowing what is appropriate to do is essential but it is not equivalent to doing that, even if there is the wish. There is definitely a distance between the statement of intentions, the knowledge and the practice- stating is one thing, knowing what to do is a different thing and actually "doing it" is yet a third one. In the particular culture the first step has probably been made, teachers are able to state the objectives of assessment rather accurately, but they do not seem to understand very well the meanings and what the implications of these objectives really are, let alone the application of their professed convictions. The small number of interviews taken is probably not enough to draw reliable conclusions. Yet one thing is pretty clear that we have a long way to go.

It still remains for us to examine what is taking place with new dynamic forms of assessment: observations and interviews, portfolios and journals, investigations and practical tasks in order to complete the picture and thence plan for developing the new "assessor teacher", redesign the teachers' programmes and supplement by inservice what is now missing from preservice education.

References

Bamboucas, M. & Troullis, G. (1993). Attitudes of teachers towards assessment. In S. Chiotakis (Ed.), The dispute of examinations and grades of students in school (pp. 200-225). Athens: Gregoris.

Cain, R.W., Kenney, P.A. & Schloemer, C.G. (1994). Teachers as assessors; A professional development challenge. In D.B. Aichele, & A.C. Coxford (Eds), Professional Development of Teachers of Mathematics, (93-102). Reston, VA: NCTM.

Gullickson, A.R. (1986). Teacher education and teacher perceived needs in educational measurement and evaluation. Journal of Educational Measurement, 23 (4), 347-354.

Pehkonen, E. (1993). What are Finnish teacher educators conceptions about teaching of problem solving in mathematics? European Journal of Teacher Education, 16 (3), 237-256.

NCTM (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: NCTM.

Stiggins, R.J. & Bridgeford, N.J. (1995). The ecology of classroom measurement. Journal of Educational Measurement, 22, (4), 271-286.

Webb, N. (1993). Assessment of standards knowledge of mathematics: Steps toward a Theory. In D. Grouws (Ed.), Handbook of Research on Mathematics Teaching and Learning. New York: Macmillan.

APPENDIX

Table giving frequencies of responses to the first four scales

	no emphasis	minor emphasis	great emphasis	major emphasis	%
What is the role of assessment in ...					
modifying instruction	20	142	316	120	73
modifying curriculum	71	247	199	77	46
finding students difficulties	1	4	86	513	99
assessing instruction	6	21	252	324	96
grading students	43	239	257	60	53
Criteria for grading include					
class participation	0	14	237	357	98
performance in class	0	2	186	416	99.5
written test scores	2	30	329	244	95
homework assignments	13	205	315	72	64
effort put forth	1	35	211	356	94
persistence and patience	5	33	253	313	94
Item format in teacher tests					
objective items (three types)	33	298	202	47	41
requiring concept definitions	47	312	169	47	38
requiring understanding	13	253	307	37	56
application to novel problems	6	70	331	174	87
application of algor. & procedures	17	80	331	135	83
similar to textbook	20	178	305	78	66
with more than one answers	31	168	297	83	67
Related instructional objectives					
application of algorithms	2	60	347	198	90
concept understanding	1	12	134	458	98
developing investigative strategies	1	21	215	369	96
problem solving	1	4	144	457	99
oral & written communication	5	36	248	345	93
developing positive attitudes	3	4	144	309	93
applications to daily life	1	36	248	407	97

Student Teachers' Conceptions of the Rational Numbers

Marcia Pinto
Departamento de Matematica
Universidade Federal de Minas Gerais
Belo-Horizonte
Brazil
e-mail: seaua@csv.warwick.ac.uk

David Tall
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL
UK
e-mail: D.O.Tall@csv.warwick.ac.uk

It is well known that students and mathematicians have idiosyncratic beliefs about the real numbers. This research reveals confusions about rational numbers in undergraduates studying to be teachers. Earlier experiences of fractions and decimals in school (such as the long exposure to finite decimal expansions and the link between π and $^{22}\!/_7$) lead to subtle misconceptions. The definition of the rational numbers is rarely used to test whether specific numbers are rational other than those explicitly given as a ratio of integers, illustrating a wider problem with the use of formal definitions in mathematics.

Introduction

It has long been known that students have idiosyncratic images of real numbers (Tall & Schwarzenberger, 1978; Davis & Vinner, 1986; Monaghan, 1986; Wood, 1992; Li & Tall, 1993; Lee, 1994; Romero i Chiesa & Azcarate–Gimenez, 1994, etc). Given children's difficulties with fractions, student difficulties with rational numbers may be expected, although the problems are often unsuspected by mathematicians. This paper reports students' responses to rational numbers in the third year of a mathematics education degree preparing primary and secondary mathematics teachers. Their earlier experiences had included schoolwork using fractions and finite decimals, a first year university programming course representing numbers as finite decimals, a second year "Sets and Groups" course defining the rational numbers in the form m/n where m and n are integers with $n \neq 0$, and a third year analysis course giving formal definitions of concepts such as limits and continuity. The small number of isolated courses may lead to less coherent integration than in specialist mathematics degrees. The students proved to have idiosyncratic "evoked concept images" (Tall & Vinner, 1981) of rationals.

An analysis of solutions of twenty students on a written assessment in analysis suggested idiosyncratic concept imagery for rational numbers. For example, in response to the question

Explain why the function $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ is discontinuous for all $x \neq 0$,

the student who achieved the highest mark in the group wrote:

> "If zero is rational, then ...
> If zero is irrational, then..."

This was indicative of subtle conflicts in students' personal imagery of rational numbers.

The role of definition in mathematics and its relationship to the concept image

The use of definitions in mathematics has two very distinct purposes. On the one hand, a concept which is already familiar to the student is given a definition to identify the concept. In this case the concept determines the definition. On the other hand, in formal mathematics the definition is used to construct the properties of the mathematical concept which it defines. In this case the definition determines the concept. This reversal from *concept→definition* to *definition→concept* is an epistemological obstacle which can cause great difficulty (see, for example, Sierpinska, 1992, p. 47). It is also an essential component of the fundamental change from elementary to advanced mathematical thinking (Tall, 1995).

From a cognitive point of view, we will be concerned that the rational number concept has a concept image in individual's mind that consists of "all the mental pictures and associated properties and processes" (Tall & Vinner, 1981). The individual's experience prior to meeting a formal definition not only affects the way in which the individual forms mental representations of the concepts, but frequently becomes manifest through the efforts to resolve problems with an inappropriate "evoked concept image".

Vinner (1991, p. 69) considered the concept definition and the concept image as two different "cells" in the cognitive structure and analysed the introduction of a definition occurring in three possible scenarios:

(i) the concept image changes to accommodate the definition;
(ii) the concept image remains as it is, the definition is forgotten or distorted;
(iii) the concept image and definition are both present but not linked together.

Our experience is that the situation is more complex. Not only are other variants possible (eg distorted concept images produced by distorted definitions, Gray & Pinto, 1995) but, more importantly, different types of connection between concept definition and concept image typically occur in a single individual. Vinner (1995) discusses how the concept image may change as a result of *pseudo-conceptual* thought, which seems on the surface to be conceptual behaviour but lacks the reflective, analytic control procedures that characterise true conceptual thinking. It is possible for each individual to have different types of concept imagery with different links to the concept definition, including:

- *informal imagery* not deduced from a definition, which may be further subdivided into imagery *consistent* or *inconsistent* with the formal theory,
- *distorted imagery* produced from a distorted personal definition or by faulty reasoning from a correct definition,
- *pseudo-formal imagery*, which may seem consistent with formal theory but is not ultimately deduced from the definition by formal reasoning,
- *formal imagery* deduced formally from the definition.

In practice it may be difficult to distinguish between these possibilities. If no definition has been given, only informal imagery is possible. (Erroneous informalities should not be classified as distorted imagery because the student has no formal definition to distort).

When a definition is given, informal ideas continue to be available and are essential to guide the individual (including the research mathematician) to formulate new ideas which can then be deduced formally. If a student remembers a (partially) correct definition, formal constructions become possible from the (correct part of the) definition, but quasi-formal and distorted imagery can also occur. The distinction between these requires investigation of the thought processes which led to their construction.

University teaching often introduces proof in an "informal manner", but this can cause great confusion in the students as to what it is that that can be assumed and what it is that has actually been "proved". This is more likely to occur where a "formal" definition includes "informal" elements, such as the rationals defined in terms of (informal) experiences of natural numbers.

"Proof" at university is often bedevilled with a mixture of formality and informality. Moreover, traditional teaching of number systems at this level usually overestimates students' informal understanding of numbers, assuming an apparent intuitiveness of the mathematical real line which does not exist in typical students. There seems to be an implicit belief that soon the learner will get the meaning of the symbolic representation by working on it formally, and so time is not spent on suitable discussion about numbers and relating this to the mathematical meaning of the real line.

Such an approach was perceived and described by one student during an interview:

> "I just took the numbers for granted really, the real numbers. When you first meet numbers and you learn what they are, you learn at some point about natural numbers and rational numbers and all of that. You feel they're just something you are supposed to know. Whether you do or not is another matter. The teacher just mentions that we're using real numbers and you never remember what they are, you just take them for granted."
>
> (Third year undergraduate student)

The long exposure to the approximate arithmetic which students encounter in secondary school is bound to exert its effects on their informal imagery. Monaghan (1986) found that students often perceive recurring decimals as dynamic and qualitatively different from finite decimals, so they are not "proper" numbers.

Our analysis will provide examples showing that a formal presentation of the subject matter and formal work with numbers did not encourage the students concerned to reconstruct their concept image from the definition of rational numbers.

Interviews and analysis of responses

Seven students selected for interview were each asked the questions:

"Can you define a rational number?"

"Can you define a irrational number?"

They were then given a list of real numbers to classify as rational or irrational. The list varied from one interview to another, but always included the number zero, square roots such as √2 and √4 which may or may not be rational, the special rational $^{22}/_{7}$ which is used in school as an approximation of π, and various finite and recurring decimals such as 0.97853, 1.41, 0.333…, 0.343232… (the last two emphasised as "recurring").

The seven responses were subdivided for study according to the student's replies to the opening questions. Three students (A, B, C) gave (subtly incomplete) formal definitions which proved to contain implicit distortions, three (I, J, K) gave explicit distorted definitions and one (student X) was unable to recall a definition. None of the students corresponded to Vinner's category (i) whose concept image changed successfully to take account of the definition.

Students offering a formal definition

Students A, B and C stated their definition close to the formal definition although all of them failed to *explicitly* exclude zero as a possibility for the denominator of a rational:

> Student A: "An irrational number is a number that cannot be expressed as two integers, one integer divided by another integer. ... And a rational number can be expressed as two integers divided by each other, *p* over *q*."

> Student B: "Well, I would define an irrational as not a rational number, and a rational number as one which we express as *p* over *q* where *p* and *q* are both integers. And it can't be expressed *p* over *q* in an irrational such as the square root of two, and π, things like that."

> Student C: "A rational number to me is a number that can be created as a ratio of two integers, I mean, this is straight Sets and Groups. Earlier on my degree course we defined all different types of number groups, and I know that one off the top of my head. Ratio of two integers, so any two integers, which could be any positive and negative whole number, the ratio of them is the rational number. And an irrational number ... is when it's not a ratio ... So—I don't know if this is helpful at all—you've got natural numbers, which are one, two, three, whatever, your integers which are negative, plus and minus one, two, plus … including zero, your rationals are integers, ratios of two integers. You might have a negative one over seven or whatever, your irrationals are non-rationals, really. So root two, … you can't express a number in terms of a ratio of two integers. "

Student C gives additional information to amplify the definition, suggesting that the definition is not seen as a generative idea from which everything can be deduced, but as a description which sets the concept in context and may require further explanation.

Students A and B give the most succinct definitions but both experience conflicts as to whether the number zero is rational or irrational:

> Student A: "... nought, I think nought's rational, but I'm not sure. My reasoning for that is because it's part of the integers, which is why I connect it with the rationals, I think."

Student A seems to be thinking conceptually, using her knowledge that integers are rationals to come to her conclusion rather than use the formal definition she has given. But there are signs of inner conflict. When asked for further explanation, she reveals that she thinks zero might be irrational because of a distorted inference that *all* rationals should have a multiplicative inverse. Because zero has no inverse, it could be irrational:

> "Why do I think it could be irrational? Because you can't divide by nought. ... Could you do nought divide by any number, will give you nought, won't it? So if you've got nought divided by *n*, for instance, you've got nought. So that way it can be expressed as a rational, but if you've got *n* divided by nought, it doesn't work." (Student A)

Student B built on a distorted conception identifying 0 with the illegal expression $^0/_0$:

> Student B: "Nought, I think that's nought over nought. ... I think you're allowed to use nought over nought where .... so it's rational, I think that's right."

When asked for further explanation, she replied:

> "... zero, isn't it ? I don't know. Maybe you can't divide by zero, so maybe you should leave ... Maybe it's an irrational. I'm not really sure whether you can have a division by zero. ... Zero divided by zero, normally you can't have zero on the bottom of a division line because it's undefined, so if it's undefined, then it wouldn't be nought either, so therefore it can't be defined as *p* over *q*. So it must be irrational.... I'm really not sure". (Student B)

The difficulties experienced by students A and B classifying the number zero may both be related to confusion over the notion of a rational as the quotient of two integers where the numerator may be zero but the denominator may not. Partially remembered this may lead to confusion as to whether the numerator or denominator may be zero. The case of zero may be further compromised when rationals are described as "positive or negative fractions or zero", treating zero as a special case distinct from familiar fractions.

Positive rationals often have informal every day meanings as fractions:

> Student A: "To represent parts of things. You use naturals for the whole things and rationals for parts of that thing. You split up into whichever number is on the bottom part, say 3 over *n*, you have to split your object into *n* parts and then 3 of those *n* parts will make your 3 over *n*."

But the following lengthy response reveals an uneasy separation between informal fractions and the technical use of rationals (as in Vinner's category (iii)):

> Student C: "I look at it as just a fraction, five eighths. ... just using it as five pieces of eight, that's just a fraction really, that's all it means to me. ... The natural number is to me the basis of maths and you probably use that every day rather, as opposed to five eighths, the fraction five eighths. To me when I would use natural numbers it would just come about every day, you are just using so many examples and uses than fractions. I would say so, just off the top of my head. ... Natural numbers to me are the simplest form of groups of numbers, there aren't many complications, you don't have to deal with negatives. ... Definitely, if you say rational number, if you went out into the street and said can you tell me a rational number, I don't know anyone, well not many people, would be able to tell you what a rational number was, in that sense. If you said give me a fraction, I mean it's like when you said what does that mean to me, that meant a fraction to me, it didn't mean a rational number, that's what it is. I'd say definitely natural numbers, more than rational and again rational is just a ... to me it's just another way of saying fractions, a fraction's more of an everyday ... more people would be able to tell you a fraction rather than a rational number. ... Again, another difference between fraction and rational number would be, say, realising that, I mean not many people would give you a negative fraction, which obviously would be a rational number, say minus five over eight. It's still a rational but they wouldn't necessarily say... I don't think that minus five over eight is a proper fraction, they would think of minus, you see what I'm saying at all? *When you said what does that mean to me, I didn't think rational number. I just thought fraction, which is a subgroup, a section of rational numbers, but that's what it means to me immediately* and now I'm putting myself in maths degree mode I'll think—Yes, that's a rational number. ... *When would I use 0.59? ... Just when you are dealing with decimal work, a part of again I suppose a fraction of unity, if you like. Or point fifty nine*. I'm trying to think now. You mean as opposed to using this, say, or you know like using, I mean fifty nine over a hundred or whether I'd use point fifty nine ..."

Instead of expanding the familiar image of fractions, a separate technical meaning has been given to rational numbers. The phrase "point fifty nine" instead of "point five nine" also indicates informal conceptions that may contain classical conflicts (such as "point fifty" being bigger than "point five", or "point eleven" being bigger than "point nine").

Although the definition given was used by all three students to *describe* rational numbers, it was not used in an operative sense to *test* whether a given number is rational, so correct responses could be pseudo-formal rather than formal. In spite of declaring the finite decimal 0.97853 correctly to be rational, Student A said:

> "I'm not sure about ... I can never remember that one."

Student B, meanwhile, classified the finite decimal 0.97853 as *irrational*, showing that she does not operate mathematically with the definition, in spite of being aware of it.

Students stating distorted definitions

Students I, J and K were not able to give a formal definition of a rational when asked to do so. Instead they attempted to describe what they thought a rational number would be, referring to their informal concept imagery. The dominance of concept image over concept definition is more transparent in these cases. For instance, students might relate the notion of rational to their experiences of finite decimals:

> Student I: "A rational number is a number that can be defined by ... it's easier to say it's not an irrational number.... An irrational number is a number with an infinite ... it cannot be defined to a finite number of decimal places. ... Yes, so a rational number would be a number that can be defined to a specific number of decimal places."

In the absence of an operative definition, Student I made decisions about the rationality of numbers in a variety of ways. For instance, she classified recurring decimals as irrationals, and the fraction $^{22}\!/_{7}$ was also considered as an irrational:

> "Because that's the way that π is represented and we've been told all through school life that π is an irrational number and can keep going for ever and ever." (Student I)

Idiosyncratic definitions often start by being stated in descriptive form, but the use, or rather lack of use, of the definition suggests an unawareness of the role of a definition in a formal context for deduction of the properties of the concept so defined. It is possible for a student to have a coherent personal definition and yet evoke other parts of the informal concept image to give a distorted image. For example, student J's definition seemed to be appropriate for some use in discriminating rationals and irrationals:

> "A rational number is a number that can be written as a fraction. For example, one is a rational number, because it can be written one over one, a half is a rational number because that's obviously one over two. An irrational number is the opposite to a rational number, you can't write it as a fraction." (Student J)

Student J correctly classified "point three recurring" as the fraction "one third", and then classified "point nine recurring" as being rational, because it was "nine over nine". The finite decimal 0.97853 and the recurring decimal 0.343232... were then classified as irrationals because in each case *she could not imagine a fraction that could represent it.*

Student K's descriptive definition of a fraction gave little opportunity for deduction:

> Student K: "A rational number? I know this one, I think. A rational number is a number that is divisible, say a whole number or a number like a half or seventeen over sixteen or something

> like that. It can be positive or negative, as opposed to irrational numbers which are square roots of some numbers."

Instead she ignored the definition and classified numbers by evoking idiosyncratic concept images. For instance decimal expansions of numbers, some finite (1.41 and 0.97853) and some recurring (0.343232...), were classified as irrationals:

> "Because when it's in decimals, I can't visualise in my head what it is, so I don't really know". (Student K)

In classifying point nine recurring correctly as a rational, she presented the argument:

> "If you rounded that up it would be a rational number." (Student K)

Further questioning revealed the pseudo-conceptual nature of the thinking based on informal imagery:

> "I don't know, it's like point nine, nine, nine ... is so close to one, but I don't know whether that makes any difference to a rational or an irrational number — being so tiny. I'm just guessing." (Student K)

Students unable to give a definition

Only one student interviewed could not formulate a definition in words, instead giving an indication of how she copes with tasks involving such knowledge:

> Student X: "I always look these up when I need to know what they are."

But she had an informal concept image of a rational number as follows:

> "I think it's rational if it can be written in terms of numbers like that" (pointing to $6/8$, a number in the given list). (Student X)

In her classification, all the decimal expressions were classified as irrationals, so she was again asked if she had a definition in mind. She replied in a manner which revealed how she avoided working with such definition:

> "Yes, but I don't know if it's right. That's what I remember. When I have to use a rational or irrational number, I've got a list of all different symbols and what the things mean; and I usually refer to that when I need to know, but it hasn't stuck yet." (Student X)

She appeared to be working with a mental list of specific examples of irrationals (such as π and $\sqrt{2}$) and rationals included numbers such as $3/5$ explicitly written in fractional form, but if the expression could not be readily converted into rational form *by her*, it was considered irrational.

Conclusion

The evidence presented in this paper reveals the rich diversity of imagery students can have relating to the concept of rational number. Although three of the seven students interviewed were able to give an (almost) satisfactory definition of the concept, none consistently used the definition as the source of meaning of the concept of rational number. Instead they used their rich concept imagery developed over the years to produce conclusions which were sometimes in agreement with deductions from the formal definition, but often were not. Whole numbers and fractions were often seen as "real world" concepts, with rationals as more technical concepts (except when positive rationals were identified with fractions). Long exposure to approximate arithmetic with finite decimals gives the latter a primacy of meaning that is sometimes erroneously identified with the fraction concept. The meaning of "rational" may also be interpreted as *the individual's ability to express the number as a quotient of two integers*, so that familiar decimals such as 0.1 or 0.333… are interpreted as rational, but less familiar decimals (particularly recurring decimals) may not be.

References

Davis, R. B. & Vinner, S. (1986). The Notion of limit: some seemingly unavoidable misconception stages, *Journal of Mathematical Behaviour*, **5** 3, 281–303.

Pinto, M. M. F. & Gray, E. (1995). Difficulties Teaching Mathematical Analysis to Non-specialists, *Proceedings of PME 19*, Recife, Brazil, II, 18–25.

Lee, B. (1994). Prospective secondary mathematics teachers' beliefs about "0.999…=1", *Proceedings of PME 18*, Lisbon, II, 185–192.

Li, L. & Tall, D. O. (1993). Constructing Different Concept Images of Sequences and Limits by Programming. *Proceedings of PME 17*, Tsukuba, Japan, III, 128–135.

Monaghan, J. D. (1986). *Adolescent's Understanding of Limits and Infinity*. Unpublished Ph.D. thesis, University of Warwick, U.K.

Romero i Chesa, C. & Azcárate Gimenez, C. (1994). An enquiry into the concept images of the continuum trying a research tool, *Proceedings of PME XVIII*, Lisbon, II, 185–192.

Schwarzenberger, R. L. E. & Tall, D. O. (1978). Conflicts in the learning of real numbers and limits, *Mathematics Teaching*, 82, 44-49.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel and E. Dubinsky (Eds.) *The Concept of Function*, MAA Notes 25, 25–58.

Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, **12** 2, 151–169.

Tall, D. O. (1995). Mathematical Growth in Elementary and Advanced Mathematical Thinking, *Proceedings of PME 19*, Recife, Brazil, I, 61–75.

Vinner, S. (1991). The Role of Definitions in Teaching and Learning Mathematics. In D.Tall (Ed.) *Advanced Mathematical Thinking*, Kluwer:Dordrecht, 65–81.

Vinner, S. (1995). The Fictitious Empire of Mathematics Education and the Behavior of its Real Subjects. In *Mathematics and Music of the Spheres*, Mathematics Education Research Centre, Warwick University, 1–18.

Wood, N. G. (1992). *Mathematical Analysis: a comparison of student development and historical development*. Unpublished PhD Thesis, Cambridge University, UK.

FOLDING BACK TO COLLECT:
KNOWING YOU KNOW WHAT YOU NEED TO KNOW

Susan Pirie — University of British Columbia
Lyndon Martin — Oxford University
Tom Kieren — University of Alberta

Abstract

Folding back is one of the key characteristics of the Pirie-Kieren theory for the growth of mathematical understanding. This paper looks at one aspect of this characteristic, that of "collecting". This phenomenon occurs when students know that they know what is needed, and yet their understanding is not sufficient for the automatic recall of usable knowledge. They need to recollect, that is to re-collect, to collect again some inner layer understanding and consolidate it through use at an outer layer in the light of their now more sophisticated understanding of the concept in question.

The notion of 'folding back'

The "Dynamical Theory for the growth of mathematical understanding" which has been developed over the last nine years and aspects of research about which have been discussed at PME meetings[1] differs from other views on the nature of this phenomenon[2] in that it characterises growth as a 'whole, dynamic, levelled but non-linear, transcendently recursive process' (Pirie and Kieren, 1991). The model developed to represent this growth contains eight potential layers of informal and formal mathematical understanding actions for a specific person and of a specified topic. These are named primitive knowing, image making, image having, property noticing, formalising, observing, structuring and inventising.[3] A diagrammatic representation of the model is provided by eight nested circles, which illustrate the fact that growth in understanding is neither linear nor mono-directional; each layer contains all previous layers and is included in all subsequent layers. Growth therefore is the result of a continual movement back and forth through the layers of knowing, as individuals reflect on and reconstruct their current knowledge.

The metaphor of recursion is used to highlight the fact that the dynamical understanding notions of a person involve states which differ in character but are self-similar. A person's current understanding action in some way acts to elaborate previous states and integrates them in the sense that they are called into current knowing actions.

A key feature of this theory is therefore the idea that a person functioning at an outer level of understanding will invocatively[4] return to an inner level. Such a shift is termed 'folding back'. When faced with a problem at any level that is not immediately solvable, an individual needs to return to an inner layer of understanding. The result of this 'folding back' is that individuals are able to extend their current inadequate and incomplete understanding by reflecting on and then reorganising their earlier constructs for the concept in question, or even to generate and create new images, should their existing constructs be insufficient to solve the problem. This the inner-level activity is not identical to that originally performed by the students, however. They now possess a degree of self-awareness about their understanding, informed by their operations at the outer level[5] and so they are effectively building a 'thicker' understanding at the inner level, to support and extend their understanding at the outer level to which they subsequently return. The theory suggests that for understanding to grow and develop folding back is an intrinsic and necessary part of the process.

---

[1] For example, Kieren and Pirie, 1992, Kieren, Reid and Pirie, 1994

[2] For a review of these see Pirie & Kieren, 1994.

[3] For a more complete description of the model see Pirie & Kieren, 1991.

[4] Invocative here is used in the context of the model to describe a cognitive shift to an inner level of

It is the purpose of this paper to distinguish what we see to be a particularly important form of folding back which we are calling 'collecting'. Folding back to collect entails retrieving previous knowledge for a specific purpose and re-viewing or 'reading it anew' in light of the needs of current mathematical actions. Thus collecting is not simply an act of recall, it has the 'thickening' effect of folding back. In what follows we give examples of collecting, distinguish it form other forms of understanding actions and discuss how teachers might act to occasion such folding back to collect. This extends the concept and uses of folding back as presented in Pirie and Kieren (1994) and Kieren, Reid and Pirie (1994) and extends our discussion of interaction and understanding (Kieren and Pirie, 1992; Pirie and Kieren, 1992).

Collecting from an inner layer

Of particular significance in the data relating to folding back is the occurrence of a number of cases where following a shift by the learner to an inner layer of understanding there has not actually been any *observable* learning activity, in the sense of any visible reorganisation or reconstruction of existing constructs nor has there been any generation of wholly new understandings. Instead of working on these existing ideas, the inner layer activity has been more a process of finding and 'collecting' an earlier construct or understanding and then using or re-reading this as useful in a new situation.

To illustrate the distinctions we are making here, we ask you to consider the following examples of three pupils tackling the question: 93 - 47 = ?

Jasmin: So, three take seven, can't do (*pause*) nine becomes eight, thirteen take seven is six (*pause*) and eight take four is four, gives forty six.

---

understanding An invocative intervention is one which promotes such a shift.

[5] Von Glasersfeld (1987) talks about 'operative awareness' and 'self-reflection' and suggests that 'what the mathematics teacher is striving to instill into the student is ultimately the awareness of a dynamic program and its execution.'

and

John: Hmm, three take seven (*pause*) hang on seven is bigger than three, I can't do it, if it was seven take three it would be OK....(*puts hand up and teacher comes over*) I can't do this 'cos seven is bigger than three so you can't take it away.

Teacher: Could you do something to the nine and the three?

John: 'erm, no, I dunno, I can't do it.

Teacher: OK then, I'll get the rods and blocks and we'll make ninety three and forty seven.

They then work with the Cuisenaire rods and use these to solve the problem.

and

Paulo: Three take seven, can't do (*pause*) no, you can do something to the nine and the three and borrow or tens it or something, lemme look. (*He opens his work book and flicks through it*). Yeah, that's it, make the nine an eight (*pause*) borrow ten so we get thirteen take seven is six. Now the other bit is eight take four is four, forty six.

The above examples are based on classroom events, and are offered to clearly illustrate different ways of thinking and working on the same problem. Jasmin has no difficulty in solving the question at all. She has the necessary understanding instantly accessible and the process she uses is essentially automatic. There is no necessity for her to fold back.

John however cannot solve the problem, it is not clear here whether he has met a question like this before but cannot now solve it or whether subtraction questions of this type are new to him. What is clear, however, is that either he does not have the necessary understanding or that his understanding is not well enough developed to allow him to use it. Instead, prompted by the teacher, he folds back to perform more image making, either to build a new image or to enhance an existing one, perhaps by working on his image for subtractions where the unit subtrahend is larger than the unit minuend. John needs to do more mathematical work at an inner layer before he will be able to build for himself an algorithm to answer the question.

When Paulo comes to tackle the question we see something very different in his thinking and working. He too cannot immediately solve the question, he does not have the automated ability to instantly use a process in the way that Jasmin did. Neither however does he fold back in the same way as John, to construct or modify an image. Paulo has an image involving the reconceptualising of the numbers that he believes will allow him to solve the question, but he needs to fold back to the level of image having in order to retrieve this image, re-view its properties in terms of the specific task at hand, and then be able to use it. There is a sense of him knowing and being aware that he has the necessary understandings but that they are just not immediately accessible and thus he needs to fold back to his more basic understanding and in some way recollect or 're-collect' it for use in his current thinking. Although initially it may appear that he has a lack of understanding of subtraction this is not actually the case; after successfully 're-collecting' the image he needs, he is able to correctly complete the question using his existing understanding of the concept. His language allows us to assume that he is not blindly applying, by rote copying, a given algorithm. He has recollected the understanding process which legitimates his subsequent action of subtraction.

It is important to note that the process of 'collecting' is a mental one. Although here it is accompanied by Paulo searching his workbook, this is not essential to the idea, it can equally be performed simply through the searching of ones thoughts. The workbook here is an aide-memoir, it is not his understandings. The major difference between this and the folding back of John is that the inner level activity of Paulo does not involve a modification of his earlier understandings. His working involves him, instead, in finding and recalling what he knows he needs to solve the problem. He is very consciously aware that this exists. He collects his inner understanding and consolidates it through intentional use.

The rest of this paper is concerned with illustrating this phenomenon of 'collecting' as it happens in the classroom for a number of different students. The examples are chosen to demonstrate some of the key features of folding back and collecting, and to indicate the varied ways in which students carry out the process and various teacher actions which facilitate it.

'Collecting' in the classroom

In the first of the following examples students are seen successfully collecting inner layer understanding and using this to continue working. In the second example the two pupils are initially less successful in doing this and provide a valuable insight into their way of thinking and their difficulties as they struggle to find and collect what they know they need.

The first extract is taken from a ninth grade lesson. The pupils are of average ability and have been set the problem of finding out the area of a segment of a circle. The teacher has simply introduced the problem by drawing a circle on the board, marking a segment and asking the pupils to find the area of it. This transcript is from when the pupils begin working:

Rosemary: There must be something on it in here. (*Pause as she flicks through her text-book*) I dunno (*in doubtful tone*), I'm looking for the area section.

Kerry: (*laughs*) Area is page a hundred and thirteen (*she turns to this page in her book*).

R: Got it.

K: There, it's half the base times the height.

R: No, (*pause*) we need ... (*pause*) It's pi r squared isn't it and erm....(*pause as she looks through book again)* Here we are, look here we are, radius and diameter so it's...it's [page] a hundred and twenty one. Circumference equals two pi r squared. No, no, no, that's wrong, two pi r. Then area equals pi r squared.

K: No, but we don't want ..

R: So, which is three hundred. (*she is working with the numbers given in the book's example then returning to the teacher's diagram which has no given dimensions)* No, that's wrong. Lets cut a quarter just to make it easy...

Here the teacher has created a situation where the pupils are able to begin working in whatever way and at whatever level is appropriate for them. However, her method of setting up the question suggests that she intends the students to begin by making an image for the area of a sector of a circle and then later to use this to find the area of the segment.

Before Rosemary begins to work at making an image for the sector of a circle, however, she folds back to her primitive knowing, searching for "something" useful and applicable to the problem. This shift appears to be self invoked, that is to say there has been no deliberate external intervention to cause her to decide to search her textbook, although obviously the question and therefore the teacher have contributed to this occurring. Although Rosemary clearly sees a need to draw on her primitive knowing and to use previous understandings in this new topic she is initially very unsure which aspects of her primitive knowing to actually fold back to and her thinking is unfocused in its nature. She says: "There must be something on it in here" without being specific. After a pause, however, she tells Kerry that she is looking for the 'area section'. She has decided that she needs to calculate the area of a circle. She finds this section in the book, intending to search for the required formula, confident that she already 'knows' it and that having re-collected it she can return to image making. She expects to be able to use her primitive knowing to continue working, in a similar manner to Paulo in our previous example. In the later stage of the extract we see that Rosemary does find the information she is searching for (both internally in terms of her own understanding and externally and physically in the text book), She 'collects' this area of a circle formula, taking it back to the level of image making where she attempts to continue working. In fact, though, she finds that she cannot automatically use her formalised rule to find a numerical answer, as the problem the teacher has posed gives no dimensions for the circle. None the less, her final statement here, "Lets cut a quarter just to make it easy" is evidence that she is now thinking about the question of finding the area of a sector. She is seeing it as a portion of the whole circle, that is to say she is constructing an image for the notion of 'sector', and she suggests that they work with a simplified initial example that will make it possible to use her recollected understandings.

The images that Kerry and Rosemary form initially are interestingly different from one another. The comment by Rosemary concerning the "area section" has a marked effect on the thinking of Kerry and as a consequence of this pupil intervention Kerry too folds back to her primitive knowing. However, her shift is more intentional, she goes directly to the concept of area of a triangle. This suggests that prior to folding back she perceived the area of the sector of a circle as being triangular in form, and attempted to collect her understanding of triangular area in order to make it possible to work with this image for a sector. For her too, the formula she needed was not automatically accessible. At the end of the extract Kerry still seems to be thinking about her area of a triangle concept and later talk does confirm that she was still thinking, at this point, of the sector as a triangle with the arc as its base.

Both students have "collected" inner understanding which they attempt to use to increase outer understanding. But the knowledge and understandings they collect result in differing ways of making an image of a sector. Both girls folded back to collect on the occasion of the given problem. Both then acted to reformulate previous understandings into an understanding of a sector. But their collecting led them to different understanding actions, just as their perceptions of the problem invoked different collectings.

The remaining extracts are taken from a teaching session with two twelfth grade pupils, Simon and Ann. They are working on calculus and within this topic, on the concept of differentiation from first principles. In order to do this the teacher has folded them back to work on the necessary primitive knowing, in this case on making an image for the concept of limits. They have already answered a number of numerical questions and are now trying to solve $\frac{h^3}{h+2h^2}$. Their initial step is simply to replace h by zero:

Ann: It's nought divided by nought...(*she writes:* $\frac{0}{0+2\times 0}$ *and* $\frac{0}{0}$*)* .
Simon: Yeah, but you're saying what's nought divided by nought? Is it nothing or is it infinity? How many nothings in nothing? Is there none or is there an infinite number?
A: Is it one? Is there one nothing in nothing?...
S: It's not one, there's not one nothing in nothing...
A; No, but if you go two by two (*pause)* over two, it's one.
S: It's one. That's different though, nothing's nothing, nothing's totally different.
A: I suppose (*pause)*. It's nothing or infinity or one, we haven't decided.
S: Its not one...

The problem here has been caused by the fact that the "h" on the bottom of the rational expression leads to a division by zero. With their present image for finding limits they are lead to replace h by zero, and they are left with a situation that they cannot solve. Their difficulty here has two aspects. Firstly their existing understanding of limits is insufficient to allow them to solve the question and secondly they do not use the necessary algebraic primitive knowing to allow them to modify their image making. Both pupils are seen folding back to their primitive knowing of arithmetic. The way in which the two pupils then work, though, differs. Simon is very much trying to retrieve a fact at this inner level, he 'knows' about infinity and zero and seems to have images for these as numbers with particular properties that differ from other numbers. Although he can call these images to mind he is left being able to state what he has recalled but unable to apply it.

With Ann the situation is different. She too folds back to her primitive knowing but she does not draw upon the same memories as Simon. Instead she folds back to an understanding of a property of division and moves out of the topic of limits to work with this property in an attempt to be able to collect her existing understanding and use it anew. She suggests its application to the particular case of "nought divided by nought". Simon is aware of the inadequacy of this notion and says 'nothing's totally different', but cannot offer an alternative idea and both students are effectively unable to proceed. At this stage the teacher intervenes:

> Teacher: Right, you can't actually give me an answer to it as it stands? In fact can you do something to that? *(pointing to original expression in "h")*, I mean what's the problem out of here is the zero on the bottom isn't it? 'Cos you don't know how to divide by zero, you don't know, as you say, how many nothings there are in something, OK? Can you do something to this? *(the original expression)* Can you simplify that in some way?
>
> Ann: You can knock them off you see...that's what we can do can't we? you can do that, you can make it h over h plus bewaaaa we've got two h squared, knock off one h squared, get h, h plus h squared. (She writes: $\frac{h \times \cancel{h \times h}}{h + 2(h \times}$ *and* $\frac{h}{h + h^2}$)
>
> Simon: Right so h plus...*(he is writing here as they work)*
>
> A: h times h, so that's *(inaudible)* times h times h. Knock off erm...
>
> S: See I've got h plus h squared over h, that's still not right...
>
> A: It's exactly the same as it used to be, erm well surely there's something we can do with these can't we? So it's still nothing divided by something, you divide it by nothing, no it's nothing...
>
> T: Are you actually happy with what was going on here (*pointing to* $\frac{h \times \cancel{h \times h}}{h + 2(h \times}$) 'cos I wasn't quite clear what was going on?
>
> T: [...] why were you able to cross that out with that?

The teacher here has recognised the problem the pupils are having and initially validates this by saying ''cos you don't know how to divide by zero.' She makes an intentionally invocative intervention to get the pupils to fold back again to their primitive knowing, but this time the teacher is able to give the intervention a more explicit focus than the printed question had provided. She asks; "Can you do something to this? Can you simplify that in some way?" The language here is a prompt to particular arithmetic/algebraic techniques. The word "simplify" seems to provide the invocative trigger for Ann and Simon. They fold back to their primitive knowing and collect from this inner layer their method and understandings of algebraic manipulation, which they proceed to work with while trying to construct an image for the notion of limits. Unfortunately it is evident that their algebraic understanding which they collect is either incomplete or inappropriate to the task at hand. Hence they will need further and other image-making including other re-collecting in order to proceed.

Implications of 'collecting' for learners and teachers

These examples have been selected to illustrate students folding back not to a reconstructive inner level activity, but to select and read anew for current use knowledge and understandings which they did not have available in algorithmic or definitional form. We call such folding back actions ‘collecting’. As is obvious from the examples, the usefulness of this collecting in on-going understanding is dependent on what is collected and how it is read into the new situation.

Interaction with materials, particularly personal notes and previously read texts, and with other students obviously affect this collecting. Thus a teacher who promotes writing about ones understanding, careful reading of texts and student discussion indirectly provides the ground for such folding back to collect. In intervening with Ann and Simon, the teacher directly promotes folding back to collect in a particular way by pointing out that algebraic simplification is needed.

Thus both the general practices of the teacher and students and specific interventions of the teacher act to invoke collecting. But it is of course, the subsequent collecting and further understanding actions of the students, rather than the intervention, which determines how their understanding in action will grow.

REFERENCES:

Pirie, S.E.B. & Kieren, T.E. (1991). A Dynamic Theory of Mathematical Understanding - some features and implications. In Annual Meeting of the American Educational Research Association. Chicago.

Kieren, T.E. & Pirie, S.E.B. (1991). Recursion and the Mathematical Experience. In L.P. Steffe (Ed.), Epistemological Foundations of Mathematical Experience New York: Springer Verlag.

Kieren, T.E. & Pirie, S.E.B. (1992). The Answer Determines the Question: Interventions and the Growth of Mathematical Understanding. In W. Geeslin and K. Graham (Eds.), Proceedings of the Sixteenth Psychology of Mathematics Education Conference. Durham, H.H.: PME. pp. 2-1, 2-8.

Kieren, T.E., Pirie, S.E.B. & Reid, D. (1994). Mathematical Understanding: Always Under Construction. Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education. Lisbon, Portugal: PME, pp. 49-56.

Pirie, S.E.B. & Kieren, T.E. (1992). Creating Constructivist Environments: Constructing Creative Mathematics. Educational Studies in Mathematics, 23(5), pp. 505-528.

Pirie, S.E.B. & Kieren, T.E. (1994). Growth in Mathematical Understanding: How can we represent it? Educational Studies in Mathematics, 26(2).

Von Glasersfeld, E. (1987). Learning as a Constructive Activity. In C. Janvier (Ed.), Problems of Representation in the Teaching and Learning of Mathematics Hillsdale: Lawrence Erlbaum.

NOUNS, ADJECTIVES AND IMAGES IN ELEMENTARY MATHEMATICS

Demetra Pitta
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL, UK

Eddie Gray
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL, UK

This paper considers the quality of images described by children at extremes of mathematical achievement. Two groups were presented with auditory and visual stimuli and asked to consider the images prompted by them. Using the grammatical notions of noun and adjective, we consider the qualitative differences in the properties and relationships identified. The similarity and the differences identified between images of concrete nouns and images of numerical nouns are illustrated. High achievers concentrate on relationships and abstract qualities of concrete, numerical nouns, icons and symbols. Low achievers highlight surface details and emphasise concrete qualities of concrete nouns and icons and see numbers as adjectives associated with concrete nouns.

INTRODUCTION

The encapsulation of arithmetical processes is regarded to be fundamental to the development of numerical concepts. Such encapsulation may be seen as the "re-concretising" of what are essentially abstract aspects of mathematics; the concept of five, abstracted from the process of counting five things, is identified through the label 'five' and the symbol 5. An action becomes an "object of thought" (Piaget, 1985, p. 49), an abstract noun associated with a numerical symbol which ambiguously represents process and concept: a procept carrying concrete and abstract ideas.

Concept formation in number development is seen to involve generalisation and abstraction from actions on physical objects. An underlying assumption in cognitive development is that eventually pedagogue and learner share common ground based upon their shared perceptual experiences. These experiences may be shared through actions on objects, for example counting, and the iconic and symbolic representations that consolidate and represent a compression of such actions. However, it is hypothesised within this paper that an understanding of the objects of action, may strongly influence the quality of encapsulation.

The paper considers children's imagery as it is exposed through verbal description. To gain insight into what children choose to communicate when asked "What comes into your mind when you hear the word... or see the icon... or symbol...", we take the view that an image is mediated by description (Kosslyn, 1980; Pylyshyn, 1973). Thus we rely extensively on language, but we realise that no precise claims can be made about the exact nature of the images. It would appear that differences between high and low achievers are not solely due to their ability to interiorise actions. We will suggest that the encapsulation of different objects–including mathematical ones–and/or the encapsulation of different components of objects may have a large part to play in mathematical achievement. Though children may look at the same thing with their "minds eye" they may see, use and manipulate things so differently. This may have consequences for their mathematical achievement (Gray & Pitta, submitted).

Identifying Mathematical Objects

We suggest that levels of abstraction are dependant upon whether or not mathematical objects are perceived to be real, and thus named as nouns, or whether they are associated with other objects and may be more adjectival in quality. Pimm (1987) suggests that such complex shifts are indications of the specialised use of mathematical ideas.

Contemporary theories of cognitive development in number may be associated with the Piagetian view that numerical concepts arise from internalised actions; processes become encapsulated as concepts. Such a notion may allow us to refer to numbers as mathematical objects and talk about them as if they were real things. Dörfler (1993) suggests that the existence of such an object may not be needed; in reviewing our own thought patterns we may not so much focus on an object but on the many relationships associated with it. However, "it" implies that such relationships are associated with a subject or object of discussion. Gray & Tall (1994) indicate that the flexibility associated with numerical relationships is crucial to the notion of proceptual thinking. To develop such thinking there is a need to recognise what the objects of mathematics are, to name them and to organise our knowledge about the relationships between them.

RESEARCH METHOD

In the belief that images created for mathematical items bear similarities with those images created for non-mathematical items, twenty four children were selected from within in a "typical" school of the English Midlands to describe images associated with mathematical and non-mathematical nouns and a range of symbols and icons. The children represented the age span 8+ to 12+, thus providing a sample size of six from each year, three 'low achievers' and three 'high achievers'. Achievement was measured by children's levels in the Standard Assessment Tasks of England and Wales (SCAA, 1994) or scores obtained within the Mathematical Concepts and Skills components of the Richmond Attainment Tests (1974). Children were interviewed individually for half an hour on at least four separate occasions over a period of eight months.

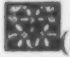
Children were presented with a range of auditory and visual items prior to mentally solving a bank of elementary numerical expressions. Here we consider the results of the auditory and visual items those within the numerical component being reported within Gray & Pitta (submitted). Responses were obtained using semi-structured interviews recorded through field notes and audio and video tapes. At each interview children were asked to talk freely about their imagery and what came to mind with each item. The auditory ones included common nouns such as "ball" and "car", and abstract nouns such as "number", "fraction" and "five". On presentation of with each item children were asked to talk about their first image. Then, at a later date, they were asked to provide a "explanation" that would help a martian understand it. The visual components, presented on a separate occasion, included symbols such as "5", 3 + 4, and , $\frac{3}{4}$ and icons such as (two quarters), (dancing man), ,(marbles) and (honeycomb).

Table 1 highlights the most powerful descriptive concepts and categories used for a discussion of the results.

	Auditory Items	Symbolic /Visual Items
1. Not Known	Unable to give meaning or any sense of recognition	Unable to give meaning or any sense of recognition
2. Associations and contextual	Child unable to pinpoint meaning–child conjectures and provides an associative theme or context.	Child unable to pinpoint meaning–child conjectures and provides an associative theme or context.
3.1 Single example	Single example that does not include symbol	
3.2 Multi examples	Several examples of the item	
3.3 Symbolic examples	Symbolic references with general characteristics: prototypical example	
4.1 Visual concrete examples.	Details of visual characteristics given. Descriptive.	Details of visual characteristics given. Descriptive.
4.2 Imaginative Extensions	Item forms basis for imaginative and/or concrete extensions.	Item forms basis for imaginative and/or concrete extensions.
4.3 Insight to abstract qualities.	Descriptions of non-visual characteristics. Insight into meaning and relationships–tend to resemble definitions	Description of non-visual characteristics. Insight into meaning and relationships–tend to resemble definitions.
5.0 Proceptual interpretation.	Emphasis on equivalence and interpretation.	Process and concept described. Examples of equivalence and interpretation

Table 1: Classification of responses to auditory and visual items

RESULTS

Because of space limitations we present just an outline of the results expanding where necessary to support later discussion.

1. Images associated with verbal items.

Figure 1 shows the classification of children's responses to auditory items and provides an indication of how they interpreted the nouns. The grouped responses, ▭ , indicate where over 25% of the total of either first image or martian explanations fall into particular classifications (12 responses for each item, number of responses in each sample 96)

- The first images of the 'low achievers' tended to be either 'association' or 'single example'. The former providing some indication of episodic memory, for example, "*people playing football*" (ball) or the recollection of personal events, for example, "*Reminds me of a friend. She was five and played with me, then she moved*" (five). Single examples included, "*my dads car, two seater*", "*a blue car*", "*figure 3*", "*figure 5*". Images classified as 'visual/description' were quite strongly linked to "five". Several of these were related to the way the five was written.

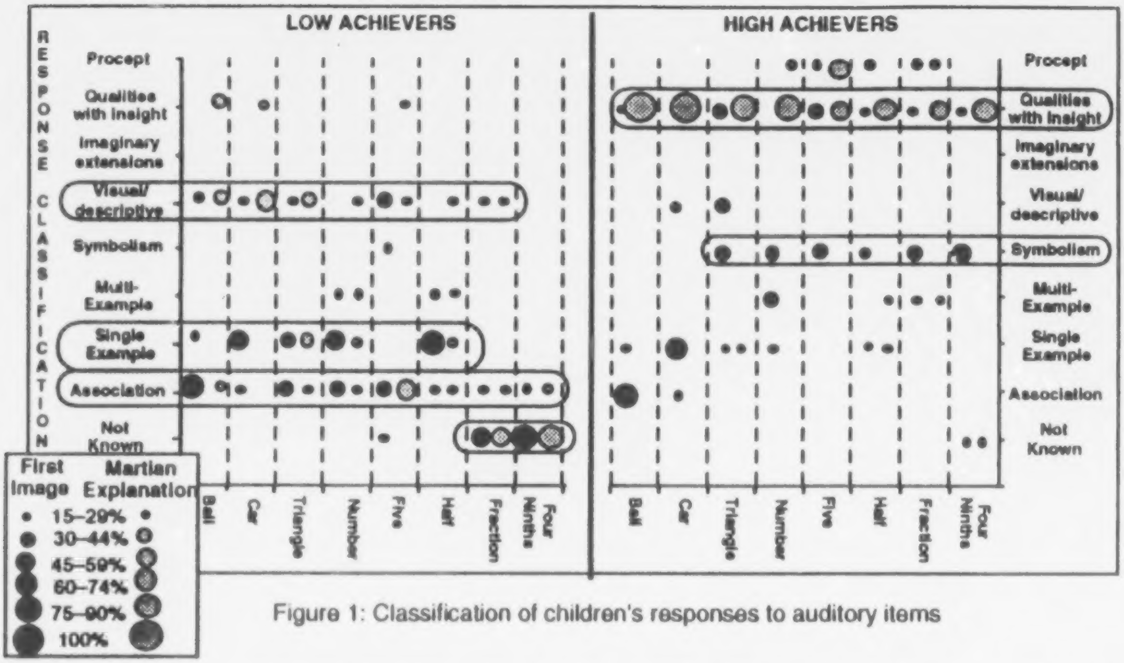


Figure 1: Classification of children's responses to auditory items

- The low achievers explanations for the martian included an extensive number of surface details for the common nouns. Phrases such as "*squares with patterns*", and "*different colours*", helped to describe ball, whilst surface details such as "*windows*", "*seats*", and "*boot*" were extensively used for 'car'. 50% of those whose initial images for the word 'ball' were associated with "*football*", provided explanations based upon the visual characteristics of a football. The abstract nouns evoked a range of responses. The dominant description for the word 'five' was "*number*" and it was strongly associated with other objects, for example, "*my sister is five*". It consistently had a concrete context placed upon it, more-so than the other items. 'Number' was either associated with mathematics or described by the visual properties of individual symbols "*some are bent and lines, some are circles*". 'Fraction' was described as "*a number on another number*", "*half a number–numbers with a line between*".
- Higher achievers also gave 'association' or 'single example' in their first responses to the common nouns but martian explanations involved a 100% shift towards descriptions based upon abstraction and non-visual qualities. Other items were extensively described in terms of 'symbolic' image or 'qualities with insight', the latter dominating the children's descriptions to the martian.

It may well be that the lower achievers provided better descriptions to the martian since the greater proportion of their information was related to visual attributes or the use of items. Frequently, particularly with the common nouns, they indicated that they would show a picture or a model. High achievers, by providing 'qualities with insight', frequently ignored the concrete and more fundamental characteristics through which the item may be recognised and focused immediately on deeper qualities.

2. Images associated with iconic representations and symbols

Figure 2 shows the classification of children's responses to visual items. The grouped responses, ▭, show where over 25% of the total of explanations fall into particular classifications (12 responses for each item, number of responses in each sample 120).

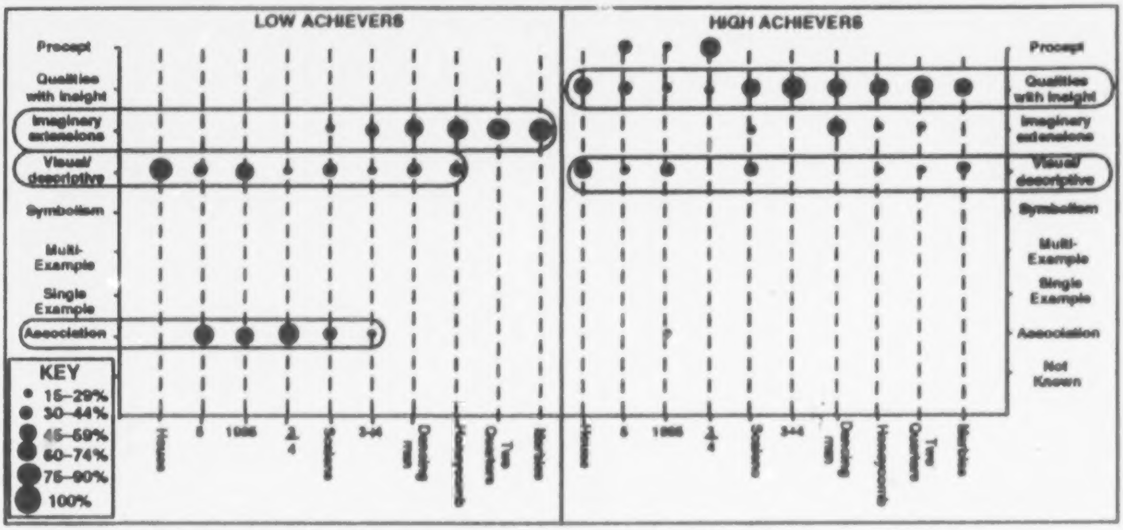


Figure 2: Classification of children's responses to symbolic and iconic presentations (Percentages)

- Low achievers had more uncontrolled reaction than high achievers to the visual items. The four iconic representations, 'marbles', 'two quarters', 'honeycomb' and 'dancing man', were strongly associated with 'imaginative description'. They were generally considered in an informal and isolated way. Detail was added as if they were "pictures out of focus". The children's efforts were directed towards inventing a story about them and giving each "picture" colour through surface details. Though the descriptions used a variety of contexts they were divorced from formal abstract vocabulary or notions such as number or shape.
- High achievers tended to look behind the icons and discuss their qualities with an insight that, where appropriate, had mathematical overtones. They recognised that icons could represent an idea. Those that could be associated with mathematics, for example, 'marbles', 'honeycomb' and 'two quarters' were described using formal mathematical language and the children provided extensive evidence of their ability to give a description of the non-visual characteristics. (Of course we are aware that these children may have anticipated the nature of "favourable" responses but as is indicated in the discussion the development of their responses changed with item difficulty). However, irrespective of age, the low achievers provided no indication that they were attempting to provide favourable responses.
- As they had shown with the numerical nouns, the low achievers extensively associated the mathematical symbols with personal detail and episodic memory. For example,

the symbol 5 was strongly associated with somebody who was five, "*someone is five years old*", "*the five on a birthday card*". The 'imaginary extensions' given for 3 + 4 indicated the inclusion of concrete examples such as "*apples*", "*sweets*" ,"*chocolate*". In essence these children need to make reference to other concrete nouns when using numerical nouns

- The high achievers consistently considered the underlying qualities of the symbols and in several instances provided examples of proceptual thinking, for example 3 + 4 was identified as "*three over four, 3/4's, 75%, three out of four*". The children appeared confident in communicating their knowledge of abstract numerical nouns.

DISCUSSION

When directed towards a particular word, icon or symbol there are an indefinite range of conclusions that the child could have drawn from the event. The children's mental representations of any concept appear to possess similar characteristics. Either the children direct attention towards its core aspects –the essential features or definition–or they direct us towards identification features through which we may recognise instances of the concept. However, the properties of the former may be different from those associated with the latter.

The similarities in the children's descriptions of imagery are remarkable both for their consistency across the range of items presented, and for the differences they display between the high achievers and the low achievers. We consider these by looking at each group of items–words, icons and symbols–separately and presenting examples which highlight the qualitative differences between the children.

Looking at the Words

First we look at the word 'five'. It tells us how old or how many of something. The low achieving children provided comments such as "*my sister is five*". In such a context "five" indicates a property of "sister" and it has conceptual characteristics analogous to the concepts of "tall" or "big" used in a adjectival way. The auditory responses of the low achievers were strongly associated with such aspects, children talked freely about "five books" or "five fingers"–the concrete objects being the books or the fingers–but did not direct attention at the core of the concept, either the counting process or the numerical concept which was the noun.

Such interpretations would seem to indicate that, in a sense, the children's understanding is on a plateau. The given property does not provide a cue to the presentation of a property at a higher level of thought. Thinking which generates the description is essentially "horizontal"–it moves from one surface feature to the next and so on. In the numerical sense it moves from one series of countable objects to the next, for example, from books, to fingers.

Essentially such "horizontal thinking" was also experienced when children considered the ball. Children who describe the colour of a ball, or who went to great lengths to describe the visual characteristics of a football were also describing features not essential to the conceptual core.

Children who describe a ball as a spherical object, a round or ovoid object used in games etc. did not focus on horizontal, frequently discrete, properties. They provided a sense that their descriptive qualities could move in a vertical plain which they traversed to provide notions of the core concept or representational features almost at will.

Looking at the Icons

Similar patterns of behaviour were identified with the iconic representations. Again, though there are an indefinite number of conclusions that may be drawn from each item, the low achievers focused extensively on visual and imaginative characteristics. These were concrete, realistic and of a similar qualityin the sense that they were seen as pictures that required colour, detail and a realistic content. High achievers concentrated on the more abstract qualities.

Again we see notions of horizontal thinking arising when the children discussed the imagery of 'two quarters', 'honeycomb' etc. They provided imaginary extensions which were essentially the same in quality i. e. "*window with curtains, window with shutters, lift doors.*"

Amongst the high achievers their was evidence that item difficulty influenced their vertical movement. The icon 'two quarters' provided evidence of such movement being in a "top down" form. Initially it triggered 'higher level' mathematical interpretations that gradually moved towards imaginative description–it was a "*shape in four quarters, half shaded....picture of a window*", or "*two out of four, half, cupboard, windows where they don't use shutters*". The more complex "honeycomb" presented evidence of a different analysis–a "bottom up" interpretation. Initially descriptive, each quality seemed to provide cues for the next level of processing –"*Hexagons, symmetrical, four light, four dark, one quarter and three quarters*".

Looking at the Symbols

The special feature of mathematics is its symbolism. The qualitative evidence indicates that children's interpretations of both linguistic and iconic stimuli have strong similarities with their interpretations of mathematical symbolism. 'Association' and 'visual/description' again dominates low achievers responses and these are accompanied by the need to concretise the symbols. This was achieved in two ways:

(i) by associating concrete items with the symbols, for example "*Three can't be divide by four because there would be a remainder. You can do it with apples though*", and,

(ii) allowing the symbols themselves to become the concrete items and then to describe the lines and curves which were their external features. This happened with several numbers to once again provide evidence of the horizontal thinking.

Amongst high achievers the abstract nature of the symbolism tended to draw upon either a bottom up analysis or, depending on familiarity, a top down one–vertical thinking. The "unfamiliar" 3 + 4 prompted responses of the former kind such as "*Fraction of some sort, not a whole number. Ah! Yes, three quarters*". 3/4 promoted both, "*Three quarters, four pieces, 3 of one sort, one of another, three quarters, one quarter, ratio 3:1....*", "*three quarters, point 75, 3 over four, three out of four, equivalence, four squares, three shaded*". This also shows that these children could describe the notions without the need to concretise them.

High achievers seem to be consistent in using the mathematical symbol as a procept and applying vertical processing characteristics in their elementary arithmetic. Similarly low achievers remain consistent in their need for concrete referents (external or internal) and horizontal thinking. For them:

- The mathematical symbol appears to be quickly translated into a concrete item, either as external referent or mental image.
- This concrete item can be changed–fingers can become sticks etc. but the quality of the item remains the same–a further example of horizontal processing.

Such transformation leads to counting processes repeated without reflection on the input/output link which, we suggest, inhibits the abstraction of, for example, the process of addition into the concept of sum.

CONCLUSION

The notion of action encapsulation lies at the heart of many contemporary theories of cognitive development in mathematics. However, actions on objects possess connotations that, we suggest, have strong implications for the quality of children's imagery. Children with different understanding of the nouns, icons and symbols associated with mathematics concentrate on different aspects of these to the point where they may attempt to encapsulate different kinds of action. The mathematics of the low achievers remains abstract; its symbols need concretising and its pictures focusing. By not understanding the nature of the abstract nouns or the symbolic nature of icons and numerical symbols we suggest they may not form the generalisations and relationships that are the hallmarks of proceptual thinking.

References

Dörfler, W. (1993). Fluency in discourse or manipulation of mental objects. *Proceedings of PME VII, Tsukuba (Japan)*, Vol. II, 145–152.

France, N., Hieronymus, A.N., & Lidquist, E.F. (1974). *Richmond Test of Basic Skills*, Windsor: NFER-Nelson.

Gray, E.M. & Pitta, D. (submitted). Number processing: Qualitative differences in thinking and the role of imagery. *Paper submitted to the reviewing procedure of XX PME, Valencia: Spain.*

Gray, E.M. & Tall, D.O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, 25, 2, 115–141.

Kosslyn, S.M. (1980). *Image and mind*. Cambridge, Mass: Harvard University Press

Piaget, J. (1985). *The Equilibrium of Cognitive Structures*. Cambridge MA: Harvard University Press.

Pimm, D. (1987). *Speaking Mathematically: Communication in the mathematical classroom.* Routlege: London

Pylyshyn, Z.W. (1973). What the mind's eye tells the mind's brain. *Psychological Bulletin*, 80, 1–24.

SCAA (1994): *Mathematics, Key Stage 2*. London: Schools Curriculum and Assessment Authority Publications.

DESIGNING A DOMAIN FOR STOCHASTIC ABSTRACTION

Dave Pratt (Mathematics Education Research Centre, University of Warwick)
and Richard Noss (Institute of Education, University of London)

Abstract

We describe the development of a computer-based domain within which children can manipulate and connect stochastic gadgets, representing everyday objects such as a dice, a coin, a lottery and a set of playing cards. These gadgets co-exist with others which may be less obviously stochastic but which are nevertheless drawn from the young child's everyday life. Observations of children interacting within this domain are allowed to shape the next iteration of software development, enabling us to gain a window onto the process by which the domain shapes the children's thinking about stochastic events, and into the software design process itself.

Introduction

The domain of probability points particularly sharply to a fundamental difficulty in mathematical pedagogy. On the one hand, it is deceptively close to everyday intuitions and experience, even language: chance encounters, random behaviours, likely occurrences. We could be forgiven for thinking that it is easy to build on these culturally embedded meanings, and that these would facilitate the transition to a mathematical way of thinking. Yet we know this is not the case; probability is a notoriously difficult topic, and it is often said that the only way for students to achieve satisfactory grades is to ignore altogether the relationship of probability to everyday notions of chance.

The explanation seems clear enough. Mathematical discourse is simply *different* from everyday discourse, and the mathematical notion of probability is a scientific, rigorous concept in contrast to the fuzzy idea of chance which pervades everyday settings. But this simple statement masks the complexity of finding a pedagogical solution. In fact, we might argue that this complexity underpins a fundamental challenge of mathematical pedagogy: to construct situations which are rich in meanings for the learner, yet which point towards the specifically *mathematical* meanings which we would like them to acquire. As individuals make their way around their social and physical world, the intellectual tools at their disposal for mathematisation and abstraction are fairly impoverished. There is no need for them: everyday, pragmatic activity is adequately served by the fuzzy linguistic tools and artefacts that have emerged in the culture over millennia. Thinking mathematically demands more: it presupposes that one has a more or less rich pool of intellectual tools at one's disposal: algebraic notation, symbolism, and so on. These are precisely the intellectual tools one has at hand if one is a mathematician, and precisely those which one lacks if one has yet to be inducted into mathematical discourse.

This puts us in a kind of pedagogical loop: We would like people to gain access and power over these tools in order so that they can make mathematical abstractions. But, in order to make mathematical abstractions, it seems that they need access to precisely these tools.

One solution to this problem, (Noss & Hoyles, in press) is to design new *domains of abstraction*, where there are tools, symbolic resources, which help to make these generalisations and abstractions. These may not be the same as mathematical ones, but neither are they the same as those we encounter in the street: they lie in a half-world between the concrete and the abstract.

Let us give an example. Why do people believe that if there is a run of 6 reds in roulette then the next spin will be likely to be black? There are good reasons. There is a sort of representativeness criterion (Tversky & Kahneman, 1974): If one walks into a party early, for example, and finds six women there, it is a good bet that the seventh will be a man. Experience leads us to believe that parties often have approximately the same number of men and women. So everyday reasoning leads to the (mostly correct) conclusion for parties, but one which is false in general: and false in the sense of roulette. Everyday settings are generally unhelpful in supporting recognition of the limitations of the representativeness criterion.

Worse still, everyday life can be downright misleading: playing roulette, it doesn't actually matter if one makes the false assumption: in reality I am as likely to win if I put my money on black as red — there is no penalty for my "wrong" action. Everything conspires to make me believe that which isn't mathematically true.

Of course, we might simply provide advice to students which goes something like this: "Ignore reality. Probability is just counter-intuitive. Always work with the definitions and standard methods.... If a trial may result in any one of *n* exhaustive, mutually exclusive and equally likely cases, and *m* of these are favourable to an event A, then the probability that A will happen as the result of the trial is measured by the quotient *m/n*." We can erect a coherent symbolic edifice which a few will understand and most will not. We know that this lack of everyday meaning is problematic, even for the few who are able to understand the formal structure. Even identifying processes as stochastic is not straightforward: dice rolling, coin tossing, the cutting of playing cards are seen by most people as stochastic in nature. However, other contexts, often drawn from the social domain, are not seen as dependent upon chance factors (Nisbett et al, 1983).

Let us give another example. In crossing the road, one tends not to think of safety as a matter of chance. By taking fairly rudimentary precautions, the chances of a successful crossing can be dramatically increased. The outcome is foremost in one's mind in the sense that a single successful crossing is *all that matters* rather than whether one acted out the optimal strategy, in stochastic terms (see Konold, 1989, for discussion on the *outcome approach*). Nevertheless, a town planner, with data about traffic accidents, may well do better to apply a stochastic model in making decisions about the location of pedestrian crossings. The choice of perspective appears to depend upon the extent to which one can exert control over the actual outcome. Restated, our pedagogic challenge is to find ways of widening the domain which is perceived by children as stochastic.

Our aims

We are trying to construct a setting in which individuals will meet the consequences of their beliefs. Our aim is to build a domain of abstraction in which the laws of probability matter, in which it is possible to *work with* these abstractions, rather than to approach the abstract as a separate domain grafted on to activity: maybe not the totality of the laws of probability, maybe not even the correct laws, but at least confronting the problem that there *are* laws of probability which are different from walking into a party or crossing the road. We want to put individual learners in situations where they can express their beliefs in symbolic (programming) form, where they can articulate the beliefs that they hold, and they are constructing those beliefs, and reconstruct them in the light of their experiences.

Method

We are developing a programming-based system[1], which will serve as a *window* (Noss and Hoyles, in press), through which we can observe children's *thinking-in-change* as they use the quasi-mathematical resources embedded in the domain and forge new meanings within the web of connections which includes both the structures in the domain of abstraction and the child's dynamically changing internal structures (see Hoyles & Noss, in press, for an elaboration of the webbing metaphor).

The notion of computer as window also applies to the process of designing the system. We wish to capture our own insights on the webbing process, by allowing our experiences of observing and interviewing the children to shape the development of the software. As each new iteration of the software is used by children, we will gain fresh insights into how these new tools influence their thinking, which will in turn re-shape the software. This process of *iterative design* lies at the heart of our methodology.

Our approach draws further inspiration from the notion that it should be easier to analyse and make sense of the design process itself as it is acted out, rather than simply examining the final product. Indeed we wish to blur the study of software design and children's construction of meaning within stochastic situations, since we see the resources available within the system and within the learner as co-existing in a symbiotic relationship.

The Software Domain

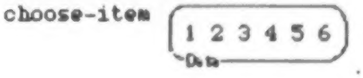
We have chosen to implement our system in Boxer (diSessa, 1985), a computational medium in which we can offer and revise resources to the learner through Boxer's reconstructible interface. At the same time, Boxer's own structures encourage learners to articulate their ideas through various modalities, including an extended Logo-type programming language.

In this paper, we will concentrate on iteration 4. The previous iteration culminated with the development of a set of "gadgets", quasi-stochastic computational mini-systems or devices which can be used as stand-alone objects but which can also be used as descriptors for other objects in the domain.

---

[1] By system, we refer both to the software, and to the social and pedagogical settings we have constructed.

All the gadgets are activated by a mouse-click and they display the result of that action. The gadgets can be opened up to reveal the basis of how they operate. So, for example, the DICE gadget (see figure 1) opens up to reveal:



```
choose-item [1 2 3 4 5 6]
```

The rest of the programming is hidden (in Boxer's CLOSET, which is in fact accessible to a more inquisitive or persistent child) so that the learner is encouraged to focus on the core mathematical elements of the action, expressed both symbolically and graphically..

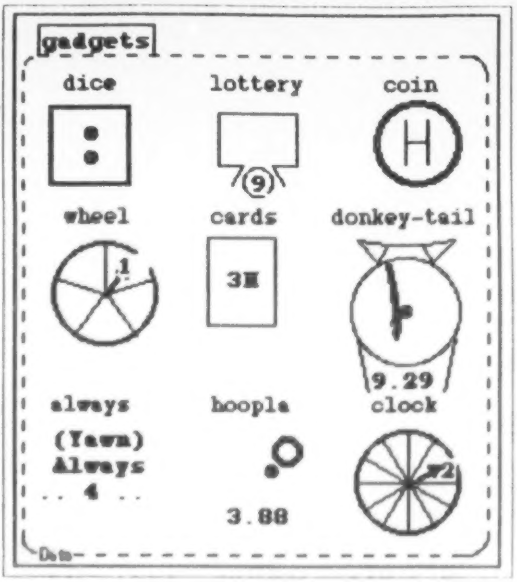


Fig 1 : The iteration 3 gadgets

The data can be modified; the child may wish, for example, to introduce a new type of dice, perhaps with bias. For example:

```
choose-item [1 2 3 3 4 4 4 5 5 5 6 6 7 8 9]
```

The clicking of the dice can also be implemented at command level: `click dice` will have exactly the same effect as clicking on DICE with the mouse.

Similarly, ROLL-A-PENNY (see figure 2) is, in one sense, just another gadget, whose operation mirrors that of the other gadgets. The Roll-A-Penny board opens up to reveal by default: `click always`. In this setting, pennies always roll to the same place on the board, but, as with the other gadgets, the operation of the Roll-A-Penny gadget could be changed arbitrarily to, say, `click dice` or indeed

```
choose-item [3 4 5]
```

.The Roll-A-Penny gadget can then be operated either through direct clicking or with the command: `click roll-a-penny`.

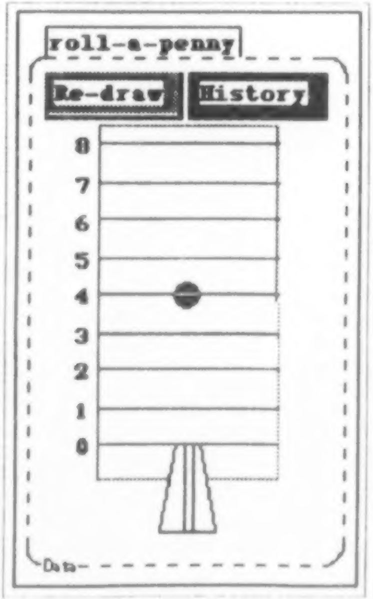


Fig 2 : The Roll-A-Penny gadget

Iteration 4 - The case of Gail and Jane

The case of Gail and Jane serves to highlight the issues which are beginning to shape our thinking about iteration 4. The two girls, age 10, have been involved in the Primary Laptop Project[2], where children have immediate and continuous access to portable computers. As a result, they have considerable experience of using computers, including many hours of using LogoWriter. However, their prior use of Boxer was limited to two sessions, of about 2 hours each[3]. Neither would be regarded by their teacher as "exceptional" in mathematics. Jane, who has been involved in the laptop project for a longer period of time, is more confident on the computer and held control of the mouse for most of the research sessions. Gail is more confident mathematically and led the discussion.

We will focus on one two-hour session in which Gail and Jane interacted with the system and responded to questions from the researcher. The session was video-taped. The discussions below are extracted from the transcript.

At the outset, the two girls were encouraged to play with the gadgets. Gail and Jane soon linked those gadgets which depended upon **choose-item** (i.e. DICE, LOTTERY, CARDS, and COIN). However, they seemed to demonstrate a different perspective when exploring some of the other gadgets. The following extract begins with Gail and Jane exploring the HOOPLA gadget, which is intended to simulate the fairground game in which the player tries to throw a hoop over a peg. It was clear that they expected to be able to control in some way where the hoop landed. The left hand column contains the researcher's questions or prompts.

	G : Why doesn't it go to the places that you point at?
That's what you feel should happen, is it?	J & G : Yes.
Do you feel the dice ... the number that the dice comes up with should be anything to do with how you click the dice?	G : The dice?
Mmm - when you click the dice, do you expect ... do you expect the number that comes up on the dice to be anything to do with how you click the mouse?	G : It won't really but it could be.
You wouldn't expect that, though?	G : No.
OK. What about the wheel? Would you expect the number that comes up on the wheel to be anything to do with how you click the mice?	G : Well, the mouse? It might depend on what you mean .... actually clicking the mouse or placing the mouse?
Yes, I'm including placing the mouse as part of	G : Yes, it does, I think.
On the wheel, you think it does. What about you Jane? Do you think ...do you think where the .... what comes up on the wheel, what number comes up, should depend on how or where you click the mouse?	J : Not really.
So you don't, but Gail does. Why do you think it should, Gail?	G : Well cos everytime I clicked there it always came there.
Did it? Do you want to try again?	

---

[2] The Primary Laptop Project is studying the effects on young children's mathematical learning when they have continuous and immediate access to portable computers. The computers are seen as part of a complex working environment, where many aspects integrate to support the children's learning. The project has just completed its third phase in which children of ages ranging from 8 to 12 took part over a period of one academic year. See, for example, Pratt (1995).

[3] The programming element of Boxer is sufficiently similar to that of Logo for this not to be too problematic.

Gail tries to click on the wheel but the spinner shows a different place.	
	G : Oh, it did before.
After a few more clicks...	
What do you think now?	G : I don't know - it might not.

There were similar episodes concerning the HOOPLA and the DONKEY-TAIL gadget; in each case the children had expected to be able to control the outcome by how they clicked the mouse but further careful experimentation showed that this did not seem to be the case. This passage shows how the girls began with a perspective in which the WHEEL, HOOPLA and DONKEY-TAIL gadgets should be controllable (and were quite different from the DICE, LOTTERY, CARDS and COIN gadgets). But experience proved otherwise; however they experimented, they were unable to make the gadgets respond causally to any action.

Later in the session, the researcher introduced the ROLL-A-PENNY gadget. After some initial exploration, Gail and Jane were challenged to make the ROLL-A-PENNY gadget work as realistically as possible. That is, to simulate the rolls of a real penny. In a previous session, the girls had played with a real roll-a-penny device and observed how the distances that a coin rolled varied slightly even when they tried to keep all the possible controlling factors the same. They had also discussed how most coins seemed to roll roughly the same distance although they noticed that a few went further and a few fell short.

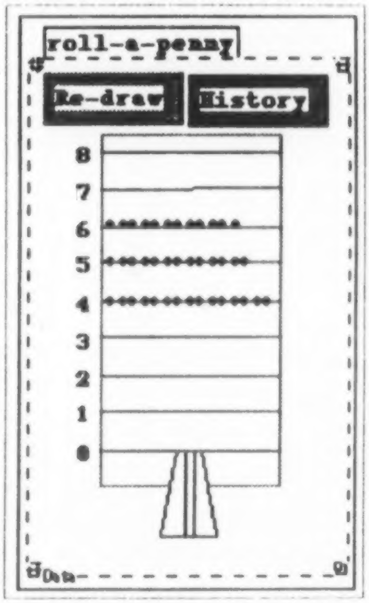


Fig. 3: A roughly even distribution after 40 rolls

The children began by modifying the DICE data box, first to include just 4 and 5 and later adding 3 and 6. After 40 rolls, the children used the HISTORY button to obtain a picture which showed a roughly equal number of each score (see figure 3).

What do you feel about that, Jane? Does that surprise you?	J : A little bit.
A little bit. In what way?	J : ...I didn't think they'd be so evenly spread. G : ..six is quite hard because that's the highest number; they are all quite hard to get because ...
Are any of them easier to get than the others, do you think?	G : No.
Are any of them harder to get than the others?	G : Not really....Well, you don't really know what it's going to do; it's just like a game....you can't really make it that realistic because, I mean .... the dice, it makes it just lucky but when you do it like on this *(referring to the real roll-a-penny device)*, it's sort of like more realistic because there's nothing like a dice or anything.

Right, but when you do the roll-a-penny, do you know before-hand exactly where it is going to go?	G : Not exactly but you've got a rough idea *Gail and Jane carry out some more rolls.* G : Well, on the real thing there was...there was a few going on 6...there were a few going on some...but there was like one part that had all - loads of them.
After a few more rolls..... How do you feel that this compares to the picture that we've had recently on the computer? What's different about this picture and the one that's on the computer?	G : Well, because we wanted the computer to look more realistic, if we put too many numbers on it could have went anywhere, but if we do this, it's, I don't know, it's on its own, we can't control it........We might be able to control that if we only put one number and then they'd all be on the same number but that would be a bit stupid.
Yes, and that would lose the realism of them going different distances.	G : Yes, and if we like put a six and a five, like, they'd all be on the same ...
The researcher encouraged the girls to think about a real dice and its six faces.	
So, you've got these six faces on the dice. What could you do to them to try and make it more like what you're doing here?	G : You could put like more of the number on one, so, like you could take the 2 off and put 6 on it instead, and you could take the 3 away and put a 5 on.

They then began to change the dice on the computer. Initially, they made the data box for the dice gadget into 4,5,6,4,5,6,7,8,3. After several more changes, they reached a point where they had four of each of 4, 5 and 6, one 7 and one 3. When they rolled the penny lots of times, they obtained a picture with many more 4's, 5's and 6's and rather less 3's and 7's.

Jane and Gail explained that there were so few 7's and 3's because they had only one of each whereas they had four of the 5's, 6's and 7's. They felt that this picture was much closer to the real results that they had previously obtained. Finally the researcher asked Gail and Jane about the issue of causality.

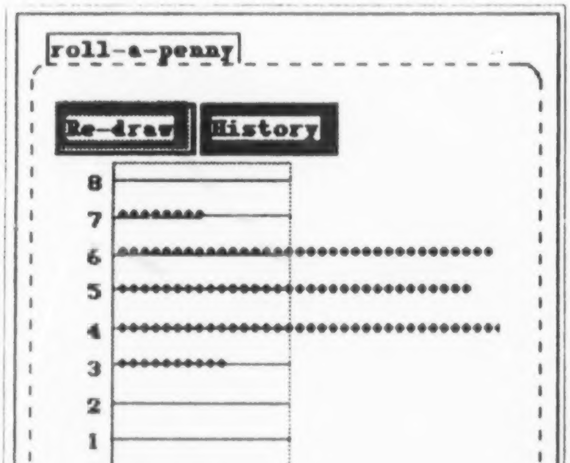


Fig. 4: Joy's and Gemma's results

How do you feel now about the extent to which you can control the real world penny compared to what we are doing on the computer?	G : Well it's a bit easier to control on the computer ...if you go back onto the dice thingy, you know on the back of it, it will tell you what the numbers are and you can take them from it, say like, if you wanted them all to be 6's, you can just write 6 on it and it will always come out with a 6.
Right, so that's a sort of control, isn't it? And what control do you have on the real roll-a-penny?	J : None, unless you push it or something.

Conclusions

Gail and Jane came to this domain expecting that they would be able to control certain gadgets and not others. They were not surprised that gadgets like the dice behaved stochastically. They knew from their everyday lives that such devices were unpredictable and uncontrollable, and this seemed to be a major criteria by which they judged whether a situation was stochastic.

However, they expected to be able to control gadgets such as the WHEEL, HOOPLA and ROLL-A-PENNY. Their everyday experiences suggested that there were factors in their control, such as how hard they threw the hoop, which affected the distance travelled, and they were surprised when they found that in this domain they could not exert control through the mouse. When they had played with the real roll-a-penny device, the coins had rolled varying distances but they explained this variation in terms of *causal* factors, how they had rolled the coins or the slope down which the coins were rolling.

In seeking to make the computer's gadget more realistic, the girls looked to introduce variation. They did this by linking the DICE gadget with the ROLL-A-PENNY gadget. Initially they found that there was too little control over where the coins landed when compared to the real device, so they narrowed down the options from which the dice was choosing. Recognising that in this domain they could easily introduce bias was a major breakthrough as they were now in a position to focus on *and control* the distances travelled by the coins. Indeed, at the end, Gail seemed to feel that the computer was easier to control than the real device, contrary to her original stance.

We are left with an emergent principle for the design of further iterations. We are convinced that a key facet of our design lies in the expressability of chance: that is, we have opened up the possibility for learners to express that which is not causal. One mechanism, which we have seen in the briefly reported data above, was the expression of unfamiliar stochastic processes (such as ROLL-A-PENNY) in terms of more familiar ones (such as DICE). At the same time, we have begun to encourage learners to focus on the biggest paradox of all: how to express - within the domain of abstraction - the structure of an aggregated set of trials, when it is impossible to predict the outcome of a single event.

References

diSessa, A., (1985). A principled design for an integrated computational environment, *Human-Computer Interaction*, 1, 1-47

Konold, C., (1989). Informal Conceptions of Probability, *Cognition and Instruction*, 6, 59-98

Nisbett, R. Krantz, D., Jepson, C. & Kunda, Z., (1983), The Use of Statistical Heuristics in Everyday Inductive Reasoning, *Psychological Review*, 90.4, 339-363

Noss, R. & Hoyles, C., (1992), Looking Back and Looking Forward, in Hoyles, C. and Noss, R. (Eds), *Learning Mathematics and Logo* , MIT

Noss, R., & Hoyles, C.,(in press), *Windows on Mathematical Meanings*, Dordrecht: Kluwer Academic Publishers

Pratt, D., (1995), Young Children's Active And Passive Graphing, *Journal of Computer Assisted Learning*, 11, 157-169

Tversky, A., & Kahneman, D., (1974), Judgement under Uncertainty: Heuristics and Biases. *Science*, 185, 1124-1131.

CORD'S STORY: AN AFRICAN AMERICAN STUDENT POINTS TO THE NEED FOR CHANGE IN COLLEGE MATHEMATICS PEDAGOGY

Norma C. Presmeg
The Florida State University

This case study of one African American student enrolled at a large State university in the USA, provides a powerful rationale for the need on such campuses to change the way mathematics has traditionally been taught using lecture pedagogy and large classes. At the time of this research, Cord was a prospective secondary school mathematics teacher, who had been deeply committed to his chosen vocation but was considering changing his career and chosen major subject because of the serious difficulties he was experiencing in university mathematics courses. Cord's story, which he told with passion, illustrates the importance of affective factors, race consciousness, and the pivotal place of positive and negative role models, in mathematical career decisions.

In her report on the demographics of access and equity in United States higher education, Bennett (1995) wrote, "Although African Americans have dramatically improved their high school completion rates, their college participation rates have declined, particularly among males" (p. 664). In the 18-24 year old age cohort, the percentage of African American males in college declined from 35.4 % in 1976 to 29.7% in 1992. Since the 1960s there has been a dramatic increase in the numbers of Black students attending predominantly White colleges and universities (ibid.). The attrition rates of Black students on such campuses are 5 to 8 times higher than the attrition rates of White students on the same campuses (p. 669). Yet the need for positive African American role models and mentors, particularly in mathematics classrooms at all levels, is as acute as ever (Banks & Banks, 1995; Mathematical Sciences Education Board, 1990). In this report of a case study of an African American male prospective teacher, Cord (pseudonym) manifested "race consciousness" in the sense of Hall and Allen (1989). Cord had a dream (Martin Luther King, Jr.) of "making a difference" in the lives of the African American high school students to whom he aspired to teach mathematics. At the time of the research, Cord was in danger of dropping out of the mathematics education program. His story dramatically illustrates some of the factors crucial in minority retention, and points to the need for change in traditional university mathematics pedagogy.

The case study of Cord is one of four case studies which were carried out in Summer, 1991. The writer was asked by the Center for the Study of Teaching and Learning to find out why many of the African American prospective teachers were experiencing difficulty in content mathematics courses. The theoretical foundation and reasons for choosing the research methodology of qualitative case studies in the investigation are expressed in Merriam (1988), for whom "A qualitative case study is an intensive, holistic description and analysis of a single instance, phenomenon, or social unit" (p. 21). The particularistic nature of this research methodology, with its inductive focus on "process, understanding, and interpretation", is eminently suited to the research question.

Four semi-structured interviews were conducted with each student over an eight week period. Three of the four students (including Cord) had completed one or more mathematics education courses taught by the author, whom they knew and trusted. Thus empathy was established quickly, and all of the students appeared to see value in the study, from which they hoped the insights attained might help their successors, even if the project was too late to spare them some hard experiences. Relevant themes of the four interviews with Cord, and their duration, are summarized as follows.

Theme	**Duration in minutes**
1. How the problem appeared to the student (and home and educational background).	30
2. Specific examples 'worked aloud' from textbooks used in the student's university mathematics courses.	20
3. Six 'word problems' from a cognitive preference measure (Presmeg's 1985 Mathematical Visuality instrument).	20
4. Beliefs - 'What is mathematics?' and final themes.	30

Cord's story

Although it is artificial to separate influences which interweave in their impact on the student's psychological functioning (Berry & Asamen, 1989), for convenience of reporting, emergent themes will be grouped around the home, the school, and the university mathematics courses.

A. The home.

The literature concurs in stressing the importance of a supportive home environment for academic success of minority students, irrespective of socio-economic status, female-headed households or other factors (Jenkins, 1989; Lomotey, 1990; Mathematical Sciences Education Board, 1990). Learning and academic achievement were valued in the two-parent middle-class home from which Cord came.

B. The school.

There were two overriding themes in Cord's description of his schooling. Firstly, there was the part played by role models in his decision to become a teacher. Secondly, this decision was influenced by experiences, both positive and negative, from the classroom. These incidents, which he had experienced or witnessed, sometimes seemed trivial in themselves, but their impact on his life was far reaching.

Although Cord came from a middle-class home, the predominantly African American elementary school he attended (a 45-minute walk from home) was in "a poor kind of neighborhood" in a large city. The junior high school he attended was "all Black", and at that time he had decided he wanted to teach business education. But in the predominantly Hispanic and White senior high he attended, there was a [White] teacher, Mrs M., of whom he said, "She is the one that really inspired me in mathematics." Along with a detailed description of how she taught particular mathematical topics, he remembered, "She would always make it so much *fun* to learn all of these kind of little acronyms there." For his five senior high mathematics courses, Cord's symbols were a B, two As and two Cs. Partly as a result of Mrs M.'s teaching, he decided to become a mathematics teacher. However, there was also an incident in the class of a different teacher, which resulted in a negative role model effect, which also influenced his decision.

Cord: "My elementary experience was very bad, and that's why I decided to become a teacher. I had some good teachers, and a lot of 'em were not! It was a negative thing that happened [in fifth grade] that inspired me to become a teacher. One of my friends had asked the teacher to explain the lesson because he did not understand where she was going or how she was explaining it. And she said, 'You know, I've explained it one time and I'm not going to explain it again, whether you all learn or not! You have to realize, I wanna get my paycheck!' And, er, when she made that comment, I thought, you know, to myself, from that day on I just, you know, I was like, *God*, I wanna be a teacher, I wanna be a *good* teacher! So, y'know, from that day I wanted to be a teacher."

Cord's passionate description was characteristic also of the way that he described his university mathematics experiences.

C. University mathematics courses.

In school, as Cord reported, "I took tests and everything and they found that I was very exceptional in mathematics." Yet in the university mathematics courses he had taken, his symbols were as follows:

Trigonometry	D, A- (course repeated)
Precalculus	C
Calculus 1	F, D (course repeated)
College Geometry	F.

He was repeating Calculus yet again, and would repeat Geometry.

Cord: "Well, right now, I'm just debating my major. I'm debating whether I want to teach any more or go to another field" [He was considering another service field, namely, public administration.]

Interviewer: "Why? Because of the math?"

Cord: "Yeah! It's just because I still have to take Linear Algebra, and, um, Statistics, and what's the other one? And Calc. 2. So I'm like, y'know, I dunno if I wanna go through that."

Interviewer: "All this heavy math in order to be a high school teacher."

Cord: "Exactly! That's where the problem comes in. I don't see the relevance for a lot of the math, but, y'know, I see it as, this department has been here for a long time, they're professionals, so apparently there's some relevancy to all this that I can't see because I'm not in the classroom. So that's how I think about it."

When questioned, Cord explained that he had taken some content mathematics courses concurrently with mathematics education courses which were taught according to a constructivist paradigm. He found a large discrepancy in the teaching methods employed. He spoke as follows:

Cord: "We sat there for two and a half hours, no, two hours [in College Geometry], listening to him lecture. It killed us! And when you take classes together, like doing, when I was doing College Geometry I was taking [the constructivist] course, Introduction to Applications for Mathematics Teachers, so it would be like, we came from [the constructivist] course, where we interacted and we just had fun, and we could apply our learning, and then we'd go to this course where we sit there for two hours and listen to this man lecture. And, y'know, we had to go through that all the time we were in [the constructivist] class, and it was just terrible! It was just, made me *miserable*!"

Cord could not see why the course, College Geometry, was required if the students were also required to take the education course, Teaching Algebra and Geometry in the High School, which was another constructivist course. He said, "Y'know, so to me that is two courses that are knocking each other." He continued as follows:

Cord: "I didn't see any relevance for College Geometry. And, with talking with people in the Department, you know, math education students even, nobody could remember. I, when I was in the course I tried to get help, and the people that took it before couldn't remember what it was about. Because, you take it and you get a grade, and you just do it!"

Interviewer: "So it's for the sake of the grade?"

Cord: "Yeah! Just because it's required."

Aspects of content mathematics courses which caused difficulty may be summarized as follows:

- fast pace of mathematics courses;
- large classes which made learning impersonal;
- quantity of content matter;
- students' excessive workload.

In the second interview, *College Mathematics*, students selected examples from a college textbook appropriate to a content mathematics course they had completed, and worked out these mathematical tasks, explaining aloud how they were thinking, what they could remember, what their impressions and feelings were. All of these students appeared to recognize the material as they paged through the books, but when they came to do examples, in detail, it was apparent, firstly, that the mathematics had been learned in a very instrumental (Skemp, 1976) or procedural way, without real understanding of the concepts (Hiebert, 1986) or of why certain procedures worked. Secondly, it was apparent that much even of this procedural learning had been forgotten. What remained was the affect or emotion which they associated with the experience of having done the course. In particular, Cord attempted five problems from a Calculus textbook, involving slope formula, implicit differentiation, a second derivative, and the quotient rule. The impression he gave in this interview was that his learning of calculus was somewhat instrumental, but that he felt confident, and was keen to show what he could do. He was currently enrolled in the course for the third time, and liked the way his current instructor taught the course, using diagrams. Some concepts were still confused; for instance he took the second and implicit derivatives as being the same thing. But in his self-confidence and increased enjoyment of the course, he gave the impression that he was coping and that he felt he would be successful that time.

Cord made strong comments regarding the negative effects of a restrictive mathematics syllabus, particularly at university level. "A lot of teachers in the math department just follow syllabus. See, I *hate* syllabus! I hate syllabus with passion! Because syllabus confines you, to so much. And it limits, you know, what the class can do. So, in all the classes that I've had before, we had a syllabus and it limits you. Like, the teacher will say, 'Well, we gotta get past this, you know, we gotta get to this point, we gotta keep going'."

The passion Cord referred to in his comments was typical of the articulations of all the students in this study, at times. The impression given was of the overriding importance of affective issues in the students' learning of mathematics at all levels, but particularly in higher education. When students learn university mathematics instrumentally, just for the grade, and then forget the course contents after the final examination, it is not just that they might as well not have taken the course. It is worse than that, because the negative affect which they associate with the experience of having taken the course, 'successfully' or otherwise, remains after the course contents are forgotten. "God, I hated all of this stuff!" said another of the students in the study, as she paged through a College Algebra textbook the contents of which she had largely forgotten.

The interview on mathematical visuality will be omitted in reporting for this paper, because the cognitive style measure of mathematical visuality, while interesting in itself, was not found to be germane to the themes which emerged in the other interviews, nor to the fact that Cord was African American. The mathematical visuality of the four students interviewed corresponded roughly to variations that could be expected in the general population.

In the fourth interview, the students were asked about their beliefs about mathematics, by means of the question, 'What is mathematics?' Cord had given this question much thought and had previously written an essay which addressed the subject. He saw mathematics as relevant to all of life, and to every other subject of the school curriculum:

> Cord: "I think of mathematics as the infinite subject; it is the subject that governs all other subjects. And when I say govern, I mean, the world cannot revolve or, the world cannot exist, without mathematics."

With understanding and eloquence, he proceeded to give examples. With regard to how he hoped to teach mathematics, *understanding* was an essential ingredient. He commented further:

> Cord: "I don't see myself as a teacher. I see myself as an educator. You can't teach anyone, you can only educate them. ... That while the students are learning from me, I'm also learning from them. And we can, you know, learn from each other. I see myself, when we're doing small groups and things, me getting into small groups, actually participating, helping, you know, exploring, and showing my students that I, too, learn new things everyday. Because a lot of times, students get a teacher and he'll know it all, the person with all of the knowledge, and they forget that they, too, can make knowledge and they just don't know how to, express it or unleash it."

It is ironic that what he aspires to in his own teaching, is in many ways a reaction to what he experienced in his university mathematics courses.

References

Banks, J. A. & Banks, C. A. M. (Eds) (1995). *Handbook of research on multicultural education.* New York: Macmillan.

Bennett, C. I. (1995). Research on racial issues in American higher education. In J. A. Banks & C. A. M. Banks (Eds), *Handbook of research on multicultural education*, 663-682. New York: Macmillan.

Berry, G. L. & Asamen, J. K. (Eds) (1989). *Black students: Psychosocial issues and academic achievement.* Newbury Park, California: SAGE Publications.

Hall, M. L. & Allen, W. R. (1989). Race consciousness among African American college students. In G. L. Berry & J. K. Asamen (Eds), *Black students: Psychosocial issues and academic achievement*, 172-197. Newbury Park, California: SAGE Publications.

Hiebert, J. (Ed.) (1986). *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Jenkins, L. E. (1989). The Black family and academic achievement. In G. L. Berry & J. K. Asamen (Eds), *Black students: Psychosocial issues and academic achievement*, 138-152. Newbury Park, California: SAGE Publications.

Lomotey, K. (Ed.) (1990). *Going to school: The African American experience.* Albany, New York: State University of New York Press.

Mathematical Sciences Education Board (1990). *Making mathematics work for minorities: Framework for a national action plan, 1990-2000.* Washington, D.C.: National Research Council.

Merriam, S. B. (1988). *Case study research in education: A qualitative approach.* San Francisco: Jossey Bass.

Presmeg, N. C. (1985). *The role of visually mediated processes in high school mathematics: A classroom investigation.* Unpublished Ph.D. dissertation, University of Cambridge.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 26-29.

ON DIALECTICAL RELATIONSHIPS BETWEEN SIGNS AND ALGEBRAIC IDEAS[1]

Luis Radford
Université Laurentienne
Ontario, Canada

Monique Grenier
Conseil de l'éducation de Sudbury
Ontario, Canada

The goal of this paper is to provide some preliminary elements for a discussion about the relationships between signs and algebraic ideas. In order to do so, in section 1, we discuss some semiotic and philosophical ideas about signs and symbols. The ideas drawn in section 1 give us a new perspective to understand some elements of the epistemological dialectic between signs and algebraic ideas (section 2). In section 3, we present some data obtained from a teaching sequence, shaped by our epistemological analysis, whose aim was to help students to evolve towards abstract symbols-ideas levels.

1. Signs, Icons and Symbols

While it seems that there is a general consensus considering symbols as a driving force of algebraic thinking, it is much less obvious to say how symbols can be used to promote algebraic ideas. Of course, we may say that algebraic ideas are actually promoted when students manipulate symbols like x, y, z. However, as a closer look at the problem shows, the modern scientific culture on which such a belief may be founded cannot suffice to sustain the thesis that algebraic ideas are automatically lagged behind symbols. Thus, in order to approach the question of how can symbols be used to promote ideas, we first need to examine, to some extent, what a symbol is.

Symbol is not seen as synonymous with sign. Let us start with the latter. What is a sign? Many linguists agree in saying that a sign is something used everyday to communicate and to signify. However, the functional status of signs do not characterize them. Mediaeval scholars used to say that a sign is something which is placed instead of something else (*aliquid stat pro aliquo*). Following this tradition, C. S. Peirce –who determinately influenced the semiotic research of the 20th century– defined, at the end of the last century, a sign as "something which stands to somebody for something in some respect or capacity" (Buchler, 1955, p. 99).

According to Peirce, a sign captures only an aspect of its *object*, this aspect is the *ground* of the sign, that is, a component of the signified (*signifié*) associated to the *object* (see Eco, 1985, pp. 34-35). On the other hand, a sign may refer to another sign by virtue of a certain abstraction. A sign, Peirce says, "creates in the mind of the person an equivalent sign, or perhaps a more developed sign" (*op. cit.* p. 99) and he calls this last sign the *interpretant* of the first sign. The *interpretant* is a new sign (see fig. 1) containing a richer cognitive content (Eco, 1988, pp. 153-54).

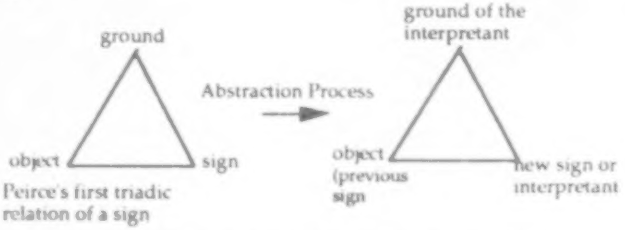
Peirce divided the signs into three categories that have been adopted by modern semiotics and are nowadays, as Eco notes (1992, p. 11), of universal usage. The three categories are: *Index*,

---

[1] This article is part of a research program supported by a grant from FCAR No. 95ER0716 (Québec) and the Research Funds of Laurentian University (Ontario).

Icons and *Symbols*. For our purposes, we only need to consider the last two. According to Peirce, "an *Icon* is a sign which refers to the object that it denotes merely by virtue of characters of its own, and which it possesses, just the same, whether any such object actually exists or not" (Buchler, 1955, p.102). The Icon reflects, then, some 'resemblance' (physically or other) with its object. In contrast, the *Symbol* is considered by Peirce as an arbitrary sign related to its object by virtue of a law or convention.

From an educational point of view, we are interested in elucidating the signifying function of symbols -a problem related to the question that drives our research and that we raised before, that is, how symbols can promote ideas. In particular, we are interested in the following set of questions which we will call Question (α):



recurrent semiotic relationship between signs

(α_1)Let s_1 be a sign of content or ground g_1 that evolves into another sign s_2 of a richer content g_2. Is the produced change caused by a modification of the first sign or by a modification of the first content? Let θ be the starting term (thus, $\theta \in \{s_1, g_1\}$).

(α_2) How does the changing process take place? More specifically:

- (α_{21}) *(external agent)* What is it that makes θ change?
- (α_{22}) *(internal dynamic)* How do the different components s_1, g_1, s_2, g_2 interact between themselves during the changing process?

The answer to this question depends on the content related to the ground and on some cultural aspects in which the signifying act takes place. The answers will also depend on what we can call the 'subject's experiential field' related to the object or concept (a field that recovers the encounters and experiences between the subject and the concept). Here, we do not need to consider this question in all of its generality; we shall circumscribe it to the case of algebra. In order to pursue our investigation let us turn our attention to the philosophical perspective about signs (something that, in contrast to semioticians, they call symbols). Even though many philosophers may agree in considering that a symbol is something placed instead of something else, as the mediaeval tradition did, the role that they give to symbols is that of accomplishing a kind of privileged transcendental 'contact' with the idea (or the *object*, to use the Peircean term) that the symbol is trying to catch (see Durand, 1964). The 'power' of symbols is precisely to allow us to make *definable* the *undefinable*, to express beyond words that which is essentially inexpressible and to «translate», beyond perceptible forms, that which is absolutely «undefinable» (Juszezak, 1985, p. 8). The symbol becomes the *epiphany* (that is, the apparition) of the unfigurable object. In the case of emerging concepts, the process of symbolization becomes an intellectually difficult adventure. In fact, how does one *call*[2] or *name* the object that does not yet have an intelligible form[3]?

To sum up our discussion, Peirce's approach to the concept of sign allowed us to raise the Question (α) that we consider relevant to teaching. The philosophical approach to the problem of symbolization sketched above makes it possible to go a step further by providing a different perspective to see the nature of the links between a *sign* and its *object* (or *idea*). The philosophical approach may not satisfactorily answer our questions insofar as it may persist in considering the *object* as an external object with a somewhat independent life from the cognizer. However, both approaches provide us a starting point to consider our question (α) from an educational perspective.

2. Some Epistemological Elements of Algebraic Signs: Naming the Un-nameable

In this section, we want to examine briefly some elements of the epistemological dialectic between signs and algebraic ideas, focusing our attention on the concept of *unknown*. In order to do so, let us remember that Babylonian scribes developed problems about geometrical figures (squares, rectangles and so on), in many of these problems, the unknown was referred to by its material name: e. g. a width, a length. No specific name was then created to designate the emerging concept. Furthermore, according to Høyrup's recent historical reconstruction (Høyrup, 1990), problem-solving procedures for many problems were guided by *figures* representing the problem (e.g. squares, rectangles) from which some parts were cut and then transferred and pasted to other sides of the figure, while other figures were added, when necessary, in order to reach a final square. The point that we need to stress is that the algebraic thinking underlying the solution of such problems was essentially *iconic*.

In contrast, Western mediaeval mathematicians used the latin word *res* and later the italian word *cosa* (the *thing*) to represent the concept of unknown. Thus, there was a specific name for the concept of unknown that applied to a great variety of problems. Even though the word *res* was taken, for a time, as synonymous with *radice* (root) -which has an obvious geometric sense- *res* and *thing* later acquired a contextual autonomy, thus becoming a *symbol* (in Peirce's sense).

---

[2] It is worthwhile to note that one of the meanings of the verb /to call/ is «to demand or ask for the presence of». Another one, even more suggestive, is «to invoke solemnly». To call, then, allows one to make something *appear*, to become intelligible to our intellect. In this context, the transcendental contact with the unattainable thing will be ensured by the symbol.

[3] Semioticians did not miss this point. For instance, in a recent book, Eco says:

"... one cannot have at one's disposal the appropriate expression until one has differentiated the content system to an appropriate degree. It is a paradoxal situation whereby the expression must be established on the basis of a non-existant content model before it can be expressed in some manner. The producer of the signs has a very clear idea of *what* he would like to say, but does not know *how* to say it..." (Eco, 1992, p. 30)

Because of the fact that the name of the concept of unknown was no longer sufficient to explain itself (in contrast to the case of icons that are self-sufficient), we find, in the 14th century, the virtuoso Maestro A. de Mazzinghi explaining the thing as "an occult or hidden quantity" –that is, a quantity whose identity will be discovered through the problem-solving process.

It is extremely difficult to retrace the conceptual movement of signs and their corresponding grounds. However, according to our research, we can suggest that the conceptual movement leading to the symbol /thing/ in the 14th century was strongly supported by an enlargement of the type of problems to solve. The square and its root did not apply to extra-geometric problems. Thus, one could be lead to extend the geometrical ideas (based on the measure of segments and surfaces) and to start considering them in a unifying numerical perspective which lead to new ideas and signs. In terms of our Question (α), the enlargement of the type of problems would play the role of an external releasing agent of the change process. The root -an iconic sign- was absorbed by an existing symbol -the thing- and this process was accompanied by an absorption and restructuration of ideas (see α_{22}).

Res and *thing* (as well as the Arabian word *shay'* for the concept of unknown) were a *catalyst* in the development of algebraic symbols. We find, in the 15th and 16th centuries, different mathematicians engaged in the search of *shorter* symbolic representations –a search mainly promoted by the need to find an easier way to carry out calculations which had become very encumbering when problems became more complex. One of these attempts was made by Piero della Francesca (*fl.* 1450); however, because of its geometric connotation, his sign system does not reach the point of independence between content and expression. Della Francesca's system can only be seen as an iconic one (see figure 2)[4]. In contrast, Michael Stifel (1544) developed a symbolic system. However, his system had the inconvenience of making it impossible to represent the operational links between the unknown and its powers (e. g. the square of the unknown is represented by a completely independent sign from the unknown, see figure 2). Attempts were also made by other mathematicians (e.g. Bombelli). The system that finally prevailed was the one introduced by Viète and reverted to and improved by Descartes. As Cajori (1919) says, Descartes' choice was arbitrary. And following Peirce, it is the condition for a system to be called symbolic. With the advent of the socially accepted Descartian symbolic system, signs moved into another status: symbols became genuine

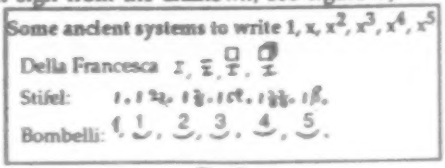


Fig. 2

---

[4] It is very interesting to note that della Francesca's efforts to construct a shorter symbolic system lead him to an iconic one, which can be seen as taking a step backwards. This failure may be seen as a symptom of the difficulties that one encounters when one tries to reach more abstract levels of thinking.

mathematical objects. The rupture with the mediaeval tradition can be seen through the fact that before algebra reached its symbolic new realm, it was not possible to pose problems using symbols only. Problems were posed in verbal form; signs were then used to translate and solve them. The recognition of the status of mathematical objects for symbols made it possible to pose problems within the symbolic system. As we can expect, the rupture with the symbolic mediaeval tradition of *res* and *thing* raised new problems. Indeed, the ground of a sign was no longer an *idea* (in its trivial sense) but a *symbol*. The recurrent chain of signs dealt with *symbolic interpretants*. As a consequence, a different way of mathematical thinking was developed –a symbolic algebraic thinking.

3. From Iconic to Symbolic Algebraic Thinking: An Example

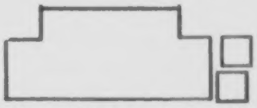
We are now going to present some experimental results of a teaching sequence for the introduction of algebra. The introduction of algebra has been studied intensively over the last years (see e.g. Rocha Falcão, 1995; Arzarello *et al.*, 1994; Bednarz and Janvier, 1994; Filloy and Rojano, 1989, Herscovics and Kieran, 1980). There are even some commercially registered manipulatives (e.g. Hands-on Equations® and Alge-Tiles™). A difference between the previous approaches and ours is to be found in the historico-epistemological basis underlying our teaching sequence. This basis allowed us to formulate our objective and to structure the sequence as follows. We wanted to help students to evolve from concrete to abstract symbols-ideas levels through a problem-solving process based on two main basic historical algebraic principles underlying the transformation of equations: (a) a *conservation–equality principle* that allows one to carry out calculations with the constant known terms; this principle also allows one to increase or decrease, in the same proportion, both members of an equation –therefore, to find an equivalent equation, 'proportional' to the preceeding equation– (see Radford, 1995a, pp. 79-80) and (b) a *restoration principle* (something historically called the rule of *al-gabr*, related to subtractive operations) allowing one to restore or fix an 'uncompleted' or 'broken' expression (see Radford, 1995b, pp. 31-32).

We carried out a three-step teaching sequence in which students had to solve some word-problems using (*i*) manipulatives, (*ii*) icons and (*iii*) symbols.

Our teaching sequence (that was video-taped) was first experienced with 6 students ,14-16 years old, from a resource center for students having difficulties and later with a regular class of Grade 9 students in a secondary school in Ontario. Working in cooperative groups of 3, they were asked to answer some word-problems classified into 6 categories. Here, we report some results from only two of them. One such category was the 'hockey card problems' (a category that can be ackwardly modelled by equations of the type $a_1x + a_2x = a_3x + a_4$ $(a_i \in N^*)$(see problem 1) and which was preceeded by another category of problems (bag-problems) related to equations of the type $a_1x + a_2 = a_3$ $(a_i \in Q^*)$. The other category that we shall consider here was the pizza-problems –a category related to equations of the type $a_1(x-a_2)+a_3=a_4(x-a_5)+a_6$ $(a_i \in N)$ (see problem 2).

Problem 1:
I have 3 envelopes each containing the same number of hockey cards plus 4 extra loose cards. I give Jacques 2 envelopes plus 1 extra card and I give Paul 1 envelope plus 3 extra cards. If I gave Jacques and Paul the same number of hockey cards, how many hockey cards are there in one envelope?

Problem 2:
André must purchase the same number of pizza slices as Louise. When they arrive at the pizzeria, they realize that the pizzas are all missing 2 slices. André buys 3 incomplete pizzas while Louise buys 1 incomplete pizza plus 4 extra slices.How many slices does one complete pizza have?


A pizza with two missing slices
figure 3

In step *(i)*, the students were provided with concrete material that we designed according to the problems. For instance, for the pizza-problems, we gave each group of students a kit containing cardboard pizzas as shown in fig. 3.

Before solving the problems, the students were familiarized with principle (a) which acquired a concrete meaning in terms of the two-plates balance that stand for the equality of expressions. Principle (b) was not directly taught. In step *(ii)* students no longer had, at their disposal, the manipulatives; they were asked to make the designs to solve the problems. The designs correspond to (perceptual) icons (see fig. 4).

The students solved, without difficulties, the problems of steps *(i)* and *(ii)*. During the problem-solving process, they referred their actions to the algebraic rules (a) and (b).

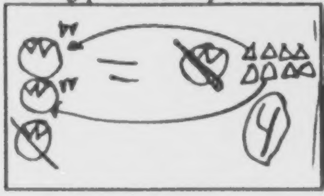
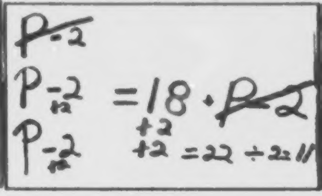
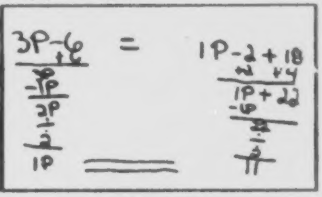


figure 4 (problem 2)


P-2
P-2 = 18 + P-2
P-2 +2 +2 = 22 ÷ 2 = 11
figure 5


3P-6 = 1P-2+18
1P+22
2P
1P
11
figure 6

Some objects of many problems in step *(ii)* were chosen in such a way that they were too long to design (e.g. 21 hockey cards). Thus, instead of drawing the objects, we asked the students to use numbers and letters. Spontaneously they used the first letter of the word. Thus, for example, pizzas were represented by the letter *p*, while envelopes were represented by *e*. Their choices coincide with a pattern mentioned by Ard 1989, p. 257:

> "Many, if not most, symbols were originally suggestive of a letter or sound. It is not accidental that *f* is the most common symbol for a function and $\vec{v}$ is the most common symbol for a vector".

In terms of the resolution of problems, the passage from step *(ii)* to *(iii)* was succesfully done. Of the group of 6 students, all, except one (to whom we shall return later; fig. 7), were able to deal with problems placed in the 'other side' of the «didactic cut» (Filloy and Rojano, 1989).

From (*ii*) to (*iii*), students kept using the *algebraic* problem-solving procedures (based on rules a and b) that they developed cooperatively in steps (*i*) and (*ii*).

Concerning the dialectic between signs and ideas, something interesting to note in step (*iii*) is the trace of (perceptual) iconic algebraic thinking on symbolic algebraic thinking. According to principles (a) and (b), to solve pizza-problems, the students eliminated known and unknown similar terms and completed the remaining uncompleted pizzas. In (*iii*), as it can be seen in fig. 5 (which refers to a problem of equation $3(p-2)=18+(p-2)$, the students eliminated first the term /p-2/ that designates a pizza with two missing slices; then they completed the two remaining uncompleted pizzas, taking care (according to principle a) to add the 2 slices twice to the member on the right of the equation (something that is placed exactly under the number 18 that represents eighteen slices of pizza). The symbol /p-2/ is seen in a *synthetic* way rather than in an *analytical* one. This means, in terms of our Question (α) (section 1), that, here, we are dealing with a phenomenon in which the sign effectively changes (from icon to symbol) while –as far as we can see– its ground did not experience a comparably significant change.

However, we were able to observe, in another instance of our teaching sequence, students moving more radically to a more abstract signs-ideas level. In fact, when solving the aforementioned pizza problem (fig. 5), a student wrote the equation /3p-6=1p-2+18/ (see fig. 6) which witnesses a first spontaneous attempt to evolve to a more abstract way of thinking. Indeed, the symbol /3p-12/ does not refer to any concrete data; it can only be understood in terms of an abstract grouping or association of the given data; in doing so, new *interpretants* (in a Peircean sense) of previous signs were constructed.

Let us now discuss one of the problems that we could detect in the passage from (ii) to (iii). Icons give a somewhat tangible spacial presence to the objects that they are representing. Thus, when students applied principles a and b, the transformed equations appeared *naturally*: the resulting equations were *there*. There was no difficulty in giving the equation any specific *spacial* configuration (see fig. 4). In contrast, when students had to use symbols, after a transformation, they had to represent the new equation (which reflects the current state of the problem-solving procedure). The new equation requires keeping track of the results of the transformation actions as well as co-ordinating, in a more abstract way, the different relationships between all of its terms. This requirement (upon which is largely based the success of a symbolic algebraic thinking) is not simple, as we can observe. In fact, fig. 7 shows the case of a student that solved completely similar problems in steps (i) and (ii) but was unable to move successfully to a symbolic algebraic thinking.

The video analysis revealed that the student loses sense of the actions undertaken. The underlying algebraic equality, which was the axis of the procedure, is lost. She is not able to give a sense to the symbols /1e+3/ (i.e. the transformed member on the left) and /5/ (i.e. the transformed member on the right).

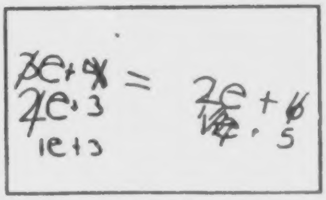


figure 7

To conclude our discussion, let us note that, in step (*iii*), the students conceived the written equation as a static support upon which one carries out the actions in order to solve the problem. This means that the written equation does not evolve in a sequential manner, line by line, as one would expect in the case of a 'competent' utilization of the algebraic language. Here, the written equation has a *heuristic* value that guides the actions of the resolution. The 'competent' utilization of the equation rests upon a code that is, like all other codes, socially constructed. It seems to us that it should be from this viewpoint that the transformations of equations be approached in the classroom. However, only after the students have had the opportunity to construct a content (i.e. a meaning) for the symbols. Otherwise, the symbols remain empty (i.e. meaningless) and, strictly speaking, cannot be symbols.

References

Ard, J. (1989) From sign to text in mathematical discourse, in *From Sign to Text*, Yishai Tobin ed., Amsterdam/Philadelphia: John Benjamins Publishing C°.

Arzarello, F., Bazzini, L., Chiappini, G. (1994) The Process of Naming in Algebraic Problem Solving, *Proceedings of the 18 PME Conference*, Vol. 2, 40-7.

Bednarz, N., Janvier, B. (1994) The emergence and development of algebra in a problem solving context: a problem analysis, *Proceedings of the 18 PME Conference*, J. da Ponte and J. Matos (eds.), Portugal: University of Lisbon, Vol. II, pp. 64-71.

Buchler, J. (1955) *Philosophical Writings of Peirce*, selected and edited by J. Buchler,New York: Dover Publications, Inc.

Cajori, F. (1919) How x came to stand for unknown quantity, *School Science and Mathematics*, Vol. 19, 698-9.

Da Rocha Falcão, J. T. (1995) A Case Study of Algebraic Scalfolding: From Balance to Algebraic Notation, *Proceedings of the 19th International Conference for the Psychology of Mathematics Education*, Universidade Federal de Pernambuco, Brazil, 2, 66-73.

Durand, G. (1964) *L'imagination symbolique*, Paris: Presses Universitaires de France.

Eco, U. (1985) *Lector in Fabula*, France: Éditions Grasset & Fasquelle.

Eco, U. (1988) *Le signe*, Bruxelles: Éditions Labor.

Eco, U. (1992) *La production de signes*, Paris: Librairie Général Française.

Filloy, E., Rojano, T. (1989) Solving Equations: the Transition from Arithmetic to Algebra, *For the Learning of Mathematics*, Vol 9, 2, 19-25.

Herscovics, N. and Kieran, C. (1980) Constructing Meaning for the Concept of Equation, *Mathematics Teacher*, 73, 572-580.

Høyrup, J. (1990) Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought, *Altorientalische Forschungen*,17, 27-69, 262-354.

Juszezak, J. (1985) *Les sources du symbolisme*, Paris: Sedes.

Radford, L. (1995a) L'Émergence et le développement conceptuel de l'algèbre (III siècle - XIV siècle), in: *Actes de la première Université d'été européenne. Histoire et Épistémologie dans l'éducation Mathématique*, F. Lalonde, F. Jaboeuf (éds.), IREM de Montpellier, pp.69-83.

Radford, L. (1995b) Before the Other Unknowns Were Invented: Didactic Inquiries on the Methods and Problems of Mediaeval Italian Algebra, *For the Learning of Mathematics*, 15 (3), 28-38.

EXPLORING UNDERSTANDING OF DATA REDUCTION

Chris Reading and John Pegg
University of New England, Armidale, Australia

There has been a growing recognition throughout Australia of the place of statistics in the mathematics school curriculum. Part of the impetus for this has been the strand status offered statistics (within Chance and Data) in ***A National Statement in Mathematics for Australian Schools****. As expected, this emphasis in the curriculum has been mirrored by similar changes in the research agenda. An important aspect of research has been the consideration of what is meant by 'statistical thinking'. This paper takes up this theme by considering students' responses to two open-ended tasks, one presents data in raw form and the other graphically. Both require a similar application of data reduction techniques. A developmental sequence of nine levels was identified and examination of the differences between the different data presentations was analysed and explained. To assist in this process the SOLO Taxonomy was employed as the theoretical framework.*

Introduction

In Australia, both state and federal educational authorities have indicated that statistical ideas should be incorporated into modern mathematics syllabuses and this should occur across the primary and secondary school years. This important change has highlighted the poor research base that exists to guide curriculum issues in statistics education. In an attempt to address this concern some researchers have begun to explore students' thinking about various aspects of statistics using a neo-Piagetian framework referred to as the SOLO Taxonomy (Biggs & Collis, 1982). This paper contributes to this trend by exploring students' responses to statistical questions involving one aspect of statistics, data reduction, and considering these in the light of the SOLO Taxonomy.

The SOLO Taxonomy has been described in detail elsewhere (see for example, Biggs & Collis, 1991; Pegg, 1992). In brief, the model comprises two aspects, modes of functioning and levels of achievement, which allow students' responses to be categorised. There are five modes and these represent a growth in abstraction: from reacting to the world by physical actions (sensori-motor); to using imaging and imagination (ikonic); to operating with second-order symbol systems such as written language (concrete symbolic); to being able to deduce general principles and work deductively (formal); to, finally, being able to challenge known theories (post formal). While these modes have much in common with those suggested by Piaget there are differences. Two are of relevance. The first concerns the placement of Piaget's "early formal" stage into the cycle of levels in the concrete symbolic mode. The second is that the earlier modes are not seen to replace subsequent modes. Instead, earlier modes continue to evolve in their own right and to support growth in later modes.

Within each mode there are a series of levels. Three levels are relevant to the work reported here. They are referred to as: unistructural - a focus on one aspect; multistructural - a focus on several aspects which are unrelated; relational - a focus on several aspects in which inter-relationships are identified. These three levels form a cycle of growth which reoccurs both within a mode and in different modes as a student responds with greater sophistication. In this case of within mode growth, the relational responses of the previous cycle are similar but not as concise as the unistructural level response of the next cycle. When different modes are explored the same pattern of cycles occur although the nature of the element upon which the level is based is different. The value of the SOLO Taxonomy lies in the depth of analysis it provides for interpreting students' responses.

Research Design

One hundred and eighty students, 30 from each of Years 7 to 12, were tested on a range of statistical questions. The students were selected randomly from each of the top, middle and bottom third of the population based on their mathematical ability. Within each year group there was a male/female balance. This paper reports on the students' responses to two questions concerning data reduction which is one of the earlier steps necessary in the process of analysing data. The aim of the questions was to present students with some data and then have them reduce that data into a more useable form. The questions were left open with no reference to any specific statistics in order to view what students perceived as necessary steps in data reduction and to allow them to use whatever facilities they had available and felt were suitable for the task. There were two forms in which the data was presented. Raw data was presented in Part I and in Part II the data was presented as a graph. This was done in order to investigate whether the form in which the data was presented influenced the way in which students reduced the data.

Analysis of Responses to Part I

The question, as it was presented to the students, is given in Figure 1 below. Upon investigating students' responses it was possible to divide them into a number of levels based on the statistical quality of the answer given. These levels were able to be grouped further based on the depth to which the response indicated the ability of the student to cope with the process of data reduction. A sample response for the each level within these groups is presented below.

Question Part I

As part of a large project which had to do with measuring and discussing the human body, one of the tasks was to measure the lengths of peoples' feet to the nearest centimetre. The results of the 29 students in the class are as follows :

26 26 26 27 27 27 27 28 28 28 28 28 28 29
29 29 29 29 30 30 30 30 30 30 30 31 32 32
33

(i) If you were asked to give a number, or numbers, (to the nearest cm.) which could be best used to represent the size of the left feet in that class, what numbers, or numbers, would you select ?
(ii) Give reasons for your selection.

Figure 1

First Group

First are those responses which dealt only with the requirements of the question. There were three Levels, coded as 0,1 and 2, observed. These responses indicate consideration of the requirements of the question with no use made of the data in formulating the response.

Level 0 These responses indicate that the requirements of the question were not understood or an answer could not be attempted.

(9209) (i) *I haven't got the faintest idea. I haven't learnt it.*

Level 1 These responses indicate an attempt at answering the question but either it is a nonsense answer or a reason is given which does not answer the question asked. For example:

(8201) (i) *Numbers.*
(ii) *Because it is a lot of numbers to choose from.*

Level 2 These responses indicate a reasonable attempt at answering the question but with no explanation of how the answer was obtained or an explanation was given that was not related to the data or question. At this stage explanations resort to personal experience rather than the data.

(8115) (i) *I would be particular, but close my eyes and put my pen down on a number.*
(ii) *It is a random selection and easy to do.*

Second Group

The second group of responses show an understanding of the question and attempt to rationalize the reduction of the data. These attempts to process the data are hampered by the lack of experiences and tools for reducing data. Again three levels were observed, coded as 3, 4 and 5.

Level 3 These responses indicate that, in attempting to justify a reasonable estimate or estimates, the reason did not refer to a feature of the data but to some feature of the question itself.

(10201) (i) *28 and 30*
(ii) *Because that would tell you how big the foot was to the closest cm.*

Level 4 These responses indicate that the data was used to obtain a reasonable estimate or estimates and an awareness that the data needed to be used to justify this answer. However, restricted experiences at data reduction result in all data being quoted as necessary in the reason.

(11111) (i) *26, 27, 28, 29, 30, 31, 32, 33*
(ii) *every childs(sic) foot is the length of one of these numbers*

At this stage there is a divergence of the responses into TWO distinct paths which appear to develop at seemingly parallel rates. These are labelled:

Path A for responses which reduce data based on measures of central tendency

Path B for responses which reduce data based on measures of dispersion.

Level 5 These responses indicate that the data could be reduced to a simple statistic. However, attempts to justify the answer usually result in an unsophisticated description of the statistic constructed. This indicates a readiness to engage in data reduction but a lack of experience and tools needed to produce a statistically sophisticated response.

Path A (10202) (i) *28 to 30*
(ii) *It is the average size for the students.*
Path B (11211) (i) *The numbers, used to represent the size of the left feet in the class, that I would choose would be 26, 27, 28, 29, 30, 31, 32, 33.*
(ii) *The reason I chose these numbers is that if I put down one of every size measured it would give you an indication of the range (from shortest to longest).*

Third Group

The final group of responses indicates an understanding of question and data. The response is given in an acceptable statistical form and the explanation attempts to relate back to the data. The three levels of responses are coded as 6, 7 and 8, with the first two being split into A and B paths.

Level 6 These responses indicate that the data could be reduced to the form of a simple statistic and the reason for making such a selection related back to the data. However, the reason stated appears to reflect a taught definition rather than a true appreciation of the data.

Path A (10215) (i) *28.86*
(ii) *because it is the mean of all the feet*
Path B (12201) (i) *26 - 33*
(ii) *That is the range.*

Level 7 These responses indicate the need to present more information by discussing more than one statistic. Reasons given still reflect the definitions of the statistics rather than discussions of properties of the data. Some responses discuss concepts which relate to statistical measures from the other path, for example a discussion of measures of central tendency may mention also some aspect of measures of dispersion. This is the stage where the two paths are beginning to converge .

Path A (12207) (i) *30 cm*
(ii) *because it worked out to be the average and also the most number of people in the class had their left foot 30 cm.*
Path B (9101) (i) *I don't really understand this question but I'll say the top one the bottom one (length) and the middle one*
(ii) *it gives a fair idea of the range in sizes.*

Responses which are not restricted to either Path A or B.

(10213) (i) *I would give the numbers 28 and 29*
(ii) *I would give these numbers as there are about the middle of the class and you can have a deviation of 3.*

Level 8 These responses indicate the use of both measures of central tendency and dispersion and the use of features of the data in an attempt to establish a link between the two.

(8215) (i) *I would select either the mode 30 or the median 29*
(ii) *The mode occurs most often and would be the most common foot length in the class, 29 is the middle of the range in foot sizes and as such should come close to most of the non-extreme values.*

The results, arranged by academic year, are presented in Table 1. These data illustrate a number of interesting points. First, there are only two students (2%) from the three senior years whose responses fall within the first group (Levels 0,1 and 2), whereas in Years 7, 8 and 9 there are a number of students (17%) who have not fully understood the question. Second, of the four Level 8 responses, 3 were in Year 12 and 1 in Year 8. Third, there is a large bulge in all years at Level 6. Last, there are three times as many students whose responses reflect Part A rather than Path B.

Table 1
Response Level and Path by Academic Year

Level	**Year**												**Total**
	7		**8**		**9**		**10**		**11**		**12**		
0	1		2		2		0		0		0		**5**
1	2		2		0		0		0		1		**5**
2	1		4		1		0		0		1		**7**
3	1		0		0		1		0		0		**2**
4	1		0		2		2		3		1		**9**
	A	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**	
5	2	1	6	0	4	0	5	0	2	1	5	1	**27**
6	16	2	10	2	13	3	13	3	14	3	9	4	**92**
7	2	0	3	0	3	1	3	0	5	0	2	0	
	1		0		1		3		2		3		**29**
8	0		1		0		0		0		3		**4**
Total	**30**		**30**		**30**		**30**		**30**		**30**		**180**

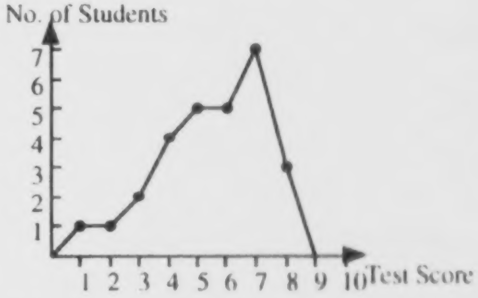
These results suggest that, for the process of data reduction given raw data, the level of response improves progressively with academic year with many students showing a tendency to feel satisfied once a simple statistic has been used in the data reduction process. Further, more students use measures of central tendency rather than dispersion to describe the data.

Analysis of Responses to Part II

The second part, II, was a general data reduction question in which the data was in a graphical form. The question was designed so that no reference was made specifically to any measure of central tendency or dispersion. However, answering this question meant that students also needed to be able to understand and interpret the graph before they were able to engage in data reduction. The question, as it was presented to the students, is given in Figure 2.

Question Part II

A teacher was interested in how students performed in a spelling test to decide whether they needed extra help. The graph below represents the scores out of 10 achieved by the 28 year 9 students in the class in the spelling test.



(i) If you were asked to give a number, or numbers, which could be best used to represent the score in the spelling test of students in that class, what numbers, or numbers, would you select ?
(ii) Give reasons for your selection.

Figure 2

When the responses to Part II were analyzed they fell into similar levels to the Part I responses, except that no Level 3 responses were identified. The hierarchy of responses includes the same nine levels, arranged into the same three groups, as the analysis of Part I responses.

The results arranged by academic year are presented in Table 2. From this data a number of interesting points can be observed. First there are less students (12%) from the three senior years who responses fall within the first group, than in Years 7, 8 and 9 where there are a number of students (20%) who have not even understood the question. Second, there was only one senior and no junior students whose responses were coded as Level 8. Third, there were no students in Level 3 and only 1 in Level 4. Fourth, there is a large bulge at Level 6 in all years. Last, there are approximately three times as many responses in Path A as in Path B. Although there are less seniors in the lower groups there is not much indication of an increase of level with academic year and irrespective of academic year many more students favour Path A than Path B.

Table 2
Response Level and Path by Academic Year

Level	Year												Total
	7		8		9		10		11		12		
0	1		4		3		2		0		0		10
1	2		3		2		2		0		1		10
2	2		1		0		5		1		0		9
3	0		0		0		0		0		0		0
4	0		0		1		0		0		0		1
	A	B	A	B	A	B	A	B	A	B	A	B	
5	2	0	6	0	5	3	2	1	4	4	4	3	34
6	16	3	13	0	11	4	13	3	18	1	8	7	97
7	1	3	1	1	1	0	1	0	1	0	3	0	
	0		1		0		0		1		4		18
8	0		0		0		1		0		0		1
Total	30		30		30		30		30		30		180

For the process of data reduction, where data are given in a graphical form, the level of response improves progressively with academic year and many more students use measures of central tendency rather than dispersion to describe the data.

Comparison of Part I and Part II

The framework appears to be adequate in explaining students' understanding as far as the basic concepts of data reduction are concerned as it was possible to allocate all responses to a level. Some students appear to have found these questions difficult with many students providing either no answer or a personal response. However, this group includes those who could not interpret or misinterpreted the graph. The slight upward shift in the trend of responses over the academic years suggests the suitability of the levels as increased understanding would be anticipated.

Testing the hypothesis that the level is independent of the part of the question yielded $\chi^2 = 16.62$ (6 d.f.) which is significant ($p < .02$) and indicates that the level into which a response is coded is strongly dependent on whether Part I or Part II of the question is being coded. Many more responses than expected were coded at the lower levels in Part II while Part I had more responses than expected in the uppermost levels. The students exhibited a higher level of understanding when the information was presented in raw form rather than as a graph. When the number of responses that were graded into each path for Parts I and II were tested, the hypothesis that the path is independent of the question part yielded $\chi^2 = 2.71$ (1 d.f.) which is not significant ($p > .1$) and indicates the path is independent of whether Part I or Part II of the question is being coded.

As far as the analysis of academic year is concerned, comparing the results presented in Tables 1 and 2 there are four noticeable trends. First, there are more junior students in the first group (Levels 0, 1 and 2) than there are senior students. Second, there are mainly senior students in Levels 7 and 8. Third, there is a large bulge in the numbers at Level 6 in every year. Fourth, there are many more Path A responses than Path B in every year. The general trend is for a slight increase in the level of performance of the students over the six academic years and a preference for measures of central tendency in the data reduction process.

The greater number of students who have responded by describing measures of central tendency (Path A) could perhaps be due to the heavy emphasis that many teachers place on mean, mode and median. This gives the students a restricted set of experiences on which to base their data reduction and in fact may force those who would naturally be inclined to follow Path B processing to follow what to them may be a less natural reasoning pattern. The large bulge of responses at Level 6 is the point at which students are using one simple statistic to describe the data. This may be due to the fact that once students have been presented with simple statistical facts at school there are limited opportunities for data exploration and so they do not have the chance to develop more advanced data reduction skills.

Despite the similarities, there are some differences between the sets of data for the two parts of the question. Three in particular show up in a comparison of the results presented in Tables 1 and 2. First, the number of senior students on Level 0, 1 or 2 is much larger for Part II (11) than for Part I (2). Second, the overall number of responses at Level 7 or 8 is much larger for Part I, (33), than for Part II (18). Third, Levels 3 and 4 have far fewer students for Part II (1) than for Part I (11).

The small number of students in Levels 3 and 4 in Part II could be due to the fact that there were more responses in the lower group for that part. There were a number of students who completely misinterpreted the graph and this made it impossible for them to formulate a sensible answer. These responses were graded in the first group and so would help to account for the larger number of students in the lower levels for Part II. This is a problem associated with understanding the graph, rather than the reduction of data and appeared to particularly be a problem with Year 10 students. The larger number of higher level responses for Part I suggests that students are better able to make detailed statistical descriptions of data when they are presented as raw data rather than as a graph.

SOLO Taxonomy Framework

The levels described earlier are now used along with the SOLO Taxonomy to create a framework which could be used to assist with the interpretation of student responses. The first group of three levels contained responses which were ikonic in character while the second and third groups represented two different cycles in the concrete symbolic (CS) mode.

The responses in the ikonic mode suggest no link could be made between the required task and any sort of symbolic representation. Such responses were mainly from students in the junior years. Within this mode, there is a framework of growth with responses similar to those in levels 0, 1 and 2. Here level 1 responses can be coded as a mixture of unistructural and multistructural levels whereas level 2 is relational.

The responses in the second and third groups have been able to link the concepts in the question to concrete experience. Every indication is that the question has been understood and the reasons given link directly to the question or the data. These responses are in the concrete symbolic mode. Within this mode two cycles became evident each containing three levels, unistructural, multistructural and relational.

The first cycle involves reducing data into a more useable form. The elements in the first cycle are the actual pieces of data themselves (data items). A relational response in the first cycle is not achieved until all data items have been considered as a functioning set and represented in a concise form. In this cycle, the unistructural, multistructural and relational levels correspond to the levels 3, 4 and 5 as outlined earlier. The level 5 responses sharing a split into two paths.

The second cycle involves appreciating that the reduction of the data creates statistic(s) which are being used to describe the features or behaviour of the data. The elements in the second cycle are the various features (or properties) of the data which statistical data reduction are trying to describe. A relational response in the second cycle is not achieved until the student is able to consider various data features and the fact that these are related when it comes to considering the overall data description. In this cycle, the unistructural, multistructural and relational levels correspond to the Levels 6, 7 and 8 as outlined earlier. These first two levels still contain the separate A and B processing paths. However, at the multistructural level there are also responses which show evidence of elements from both paths.

Conclusion

It is possible to develop a hierarchy of levels to categorize responses to questions involving data reduction and these categories can be interpreted using the SOLO Taxonomy. Also, it appears that there are two possible paths of reasoning in the second and third groups (first and second cycle CS mode, respectively) and the form in which the data is presented may influence a student's choice of method of data reduction. Overall, the understanding of data reduction appears to be better when the data is presented in raw form rather than as a graph. In SOLO terms, many students, particularly those who responded in the first cycle (CS mode) with data in raw form, found the additional cognitive load associated with interpreting the graph to great and responded to Part II with responses of the ikonic mode. Students who responded in the second cycle (CS mode) did not, in general, have this problem. Students are far more likely to reduce the data using measures of central tendency than dispersion irrespective of the form of data presentation.

References

Australian Education Council (AEC) (1991). *A National Statement on Mathematics for Australian Schools,* Carlton, Victoria: Curriculum Corporation.

Biggs, J. & Collis, K. (1982). *Evaluating the Quality of Learning: the SOLO Taxonomy.* New York: Academic Press.

Biggs, J. & Collis, K. (1991). Multimodal learning and the quality of intelligent behaviour. In H. Rowe (Ed.), *Intelligence, Reconceptualization and Measurement* (pp. 57-76). New Jersey: Laurence Erlbaum Assoc.

Pegg, J. (1992). Assessing students' understanding at the primary and secondary level in the mathematical sciences. In J. Izard & M. Stephens (Eds), *Reshaping Assessment Practice: Assessment in the Mathematical Sciences Under Challenge* (pp. 368-385). Melbourne: Australian Council of Educational Research.

"Wouldn't It Be Good If We Had A Symbol To Stand For Any Number": The Relationship between Natural Language and Symbolic Notation in Pattern Description

Dr Ted Redden

University of New England

Armidale, Australia

Discussion on the use of number patterns to provide a context for the development of early algebraic concepts and notation often emphasises the role of language as a significant step in the developmental sequence. This paper reports on a set of responses collected from a large sample of 10 to 13 year old children. The responses were stimulated by requests to generalise number patterns in a variety of ways. The data was coded in a qualitative way prior to being subjected to a model building process using multi -way frequency analysis and the calculation of parameter estimates measuring the strength and direction of the association between response categories. The findings indicate a significant association between the natural language descriptions and the symbolic notation used by children. Further, teachers need to be aware that only one group of natural language descriptions appears to lead to the development of algebraic notation.

The need to contextualise the initial instruction in algebra has been the subject of considerable discussion in recent years (Rojano & Sutherland, 1993; Mason et al., 1985). Several such contexts have been considered but those that seem to have been the subject of considerable discussion include number pattern environments and technological environments such as spreadsheets. It seems inherently sensible to teach the language of generality to children by providing them with something to generalise.

Mason (1985, 1992) popularised an approach to developing algebraic thinking through the expression of generality in contexts that involve some form of number pattern. This approach has been adopted and developed by others (Pegg and Redden, 1990a; Pegg and Redden, 1990b; Board of Studies, 1989; Romberg, 1989; Australian Education Council, 1991). While the approach has had considerable support, the research evidence (Pegg and Redden, 1990c; Arzarello, 1991; Redden, 1993; Redden, 1995a; MacGregor and Stacey, 1993) that has been reported on this approach has identified a range of response types in children's attempts to describe number patterns, indicating a variety of conceptualisations of those patterns.

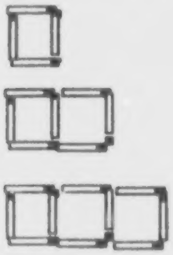
The Vygotskian perspective (1986) of the role of language in concept development is supported by Mason et al (1985) and Pegg and Redden (1990), who have emphasised the role of natural language in the use of patterns for developing algebraic notation. If the use of natural language is central to this development, it would seem reasonable to propose that there is a relationship between the natural language children use to describe patterns and their ability or willingness to use algebraic notation. Hence this paper investigates the research question:

> Is there an association between the categories of pattern description using natural language and the categories of symbolic language used?

Method

To investigate this question 1435 children aged 10 to 13 were presented with 4 number pattern stimulus items and were asked four questions about each item. The first question was designed to investigate comprehension of the stimulus item. The second question was designed to investigate the children's perception of the number pattern, while the third question explored their ability to apply their rule beyond a countable example. Finally, the last question investigated the children's readiness to express their natural language using the formal symbolism of mathematics. The precise wording of these four questions changed with both the pattern context and the mathematical background of the children. One of these stimulus items is presented here as Figure 1.

Here are some chains of Matches



(a) What is the next term in the pattern?

(b) Describe a general rule for the pattern in natural language

(c) Calculate the value of an uncountable term (eg. n=80)

(d) Write their rule in the symbolic notation of mathematics

Figure 1
A number pattern Stimulus Item

Approximately half the children (Years 5 and 6) had no formal exposure to school algebra while the older children (Years 7 and 8) had been exposed to some algebra instruction, which may have included the use of number patterns as a contextual vehicle.

This paper reports the data associated with just one of the four stimulus items (see Figure 1). Similar findings were associated with the other three stimulus items (Redden, 1995).

The focus of this analysis is the coded responses of the 1435 respondents. This coding process took steps to establish both reliability and validity. The issue of reliability was addressed by calculating coefficients of inter-coder and intra-coder reliability. A coefficient of 90% was set as the criterion for adequate reliability in the process. Validity of the coding process was established by comparing coding categories with the reported categories of other researches using related items (O'Brien, 1991; Macgregor and Stacey, 1993; Ursini, 1991; Meira, 1990; Erickson, 1988).

Five major categories of natural language were identified. Table 1 provides the category name, descriptor, an example of a response and the frequency of responses in that category Table 2 presents a similar set of data for the use of symbolic notation categories.

To investigate of the relationship among the various component codes the technique of multi-way frequency analysis available on the SPSS platform as Log-Linear Analysis (Norusis, 1990) was used.

Table 1
Natural Language Categories and Frequencies (n=1435)

Code	Descriptor	Example	Frequency
No Attempt	No attempt was made to answer the question		360
Inappropriate	This group indicated a failure to understand the question or repeated the question	How many wheels do you need?	95
One Example	This group gave a value for a specific example rather than a general description	To make 5 squares you need 16 matches	306
Successive Description	This group made use of the dependent variable only. They tried to calculate a term in the series given the previous term	Add three every time	294
Function	This group described a relation between the independent and dependent variables	The number of matches is the number of squares times three plus one	380

Table 2

Symbolic Notation Categories and Frequencies (n=1435)

Code	Descriptor	Example	Frequency
No Attempt	No attempt was made to answer the question		557
Operation Symbols	Attempts at symbols were restricted to those commonly used in arithmetic	$4 \times 5 = 20$	279
Arbitrary use of letters	These responses indicated a knowledge that letters were used in mathematics but reflected little knowledge about their correct use in algebra	$x + y$ *or* $a = 4 \times 2$	182
Iterative Description	Symbols reflect the repeated adding of a constant term	$4 + 3'$ *or* $y = 4 + 3 + 3 + 3.....$	140
Algebra	Successful use of algebraic ideas	$x = 4 + 3(n - 1)$	277

The process of model-building involved taking four variables into the model (the above two together with school year and responses to question (C) in Figure 1) and removing complex interaction terms in a step down selection strategy until a parsimonious model was identified that adequately represented the data. A full description of the process and underpinning assumptions is reported elsewhere (Redden, 1994; Redden 1995). The resulting model provided a conceptual framework for the analysis of significant relationships within the data. In particular, log-linear analysis provides parameter estimates describing the strength and direction of associations between the categories of the variables being considered. Even with the large sample of 1435 children, limitations of expected cell frequencies required restricting the number of categories for natural language and symbolic notation to four each. As a result of carefully considering similarity between categories the decision was made to combine the No Attempt and Inappropriate categories in the Natural Language component, and to combine the Arbitrary use of Letters and Iterative Description categories in the Symbolic Notation component.

Results

The question being investigated in this paper requires us to focus on the association between Natural Language and Symbolic Notation. From the data presented in Table 3 it can be seen that the null hypothesis:

> that Natural Language and Symbolic Notation are independent

must be rejected due to the partial associations for the interaction being significant ($p<0.05$, df=9). Therefore there is a significant association between the responses to Natural Language and the Symbolic Notation responses. A more detailed picture of this relationship can be gleaned by considering the contingency table and the associated parameter estimates. This information is presented as Tables 3 and 4.

Table 3
Contingency Table Comparing Natural language and Symbolic Notation

Natural Language	**Symbolic Notation**				
	No attempt	Operation Symbols	Arbitrary and Iterative notation	Algebra	Totals:
No attempt/ inappropriate	374	41	26	14	455
One Example	93	149	56	8	306
Successive Description	53	30	181	30	294
Function	37	59	59	225	380
Totals:	557	279	322	277	1435
Partial association=683.28					p<0.05

The frequency data is presented in graphical form in Figure 2. The pronounced saddle shape indicates the changing level of association between pairs of categories.

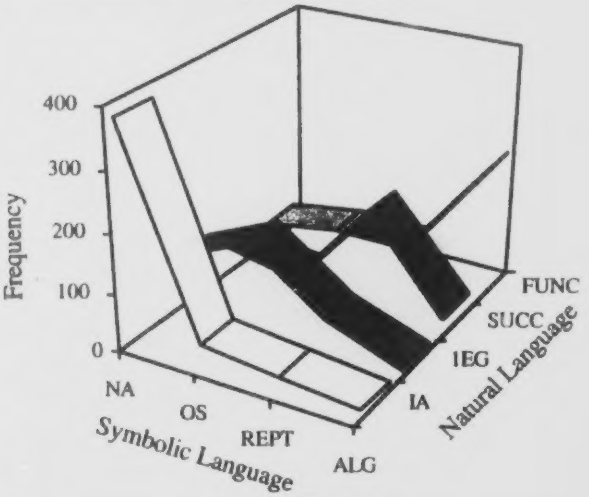


Figure 2
Response Surface of Natural Language and Symbolic Notation

The strong positive associations indicated by the parameter estimates of Table 4 are represented by the high points on the ridge of the responses surface.

The associated Z scores for the parameter estimates are also presented in Table 4 and indicate a number of pairs of association significant at the $p<0.0005$ level. (The very conservative alpha value of 0.0005 ($Z=3.291$) was taken, in order to guard against a rapidly escalating type 1 error rate that might arise from the large number of comparisons being made).

Table 4
Parameter Estimates and Z Values of Natural language and Symbolic Notation Comparisons

Natural Language	**Symbolic Notation**			
	No attempt	Operation Symbols	Arbitrary and Iterative notation	Algebra
No attempt/ inappropriate	1.358* z=14.104	0.008 z=0.072	-0.416* z=-3.748	-0.950* z=-8.162
One Example	-0.287 z=-2.193	0.829* z=7.322	-0.384 z=-2.869	-0.158 z=-1.353
Successive Description	-0.677* z=-4.952	0.089 z=0.744	0.993* z=10.555	-0.405* z=-3.787
Function	-0.393 z=-2.157	-0.926* z=-4.257	-0.192 -1.295	1.512* z=12.905
* Significant at p<.0005				

The high ridge on the response surface in Figure 2 reflects the significant positive associations between the pairs of categories **(IA, NA)**, **(1EG, OS)**, **(SUCC, REPT)** and **(FUNC, ALG)**. Further, the only significant positive associations found among all four stimulus items lay on this ridge. This ridge represents the diagonal of the contingency tables going from top left to bottom right. Conversely to the above positive associations, the significant negative associations all lie off the diagonal identified.

The direction and strength of the associations of categories between the Natural Language component and the Symbolic Notation component indicate a consistent relationship between the natural language of pattern description and the symbolic notation to represent that language. However, there was a design issue that may have interfered with the clarity of this relationship. It will be recalled that Years 5 and 6 had had no experience with algebra and that Years 7 and 8 had had some introductory algebraic experience. It seemed reasonable to pose the question of whether the associations identified above would be similar among just the Year 7 and 8 students who had had experience of learning algebra.

Previously, the three way effect of Natural Language by Symbolic Notation by School Year was not included in the log-linear model, with the implication that there is a similar pattern of association for Natural Language by Symbolic Notation among the Year 7 and 8 children as there is for the children in the whole sample. To investigate this a confirmatory log-linear analysis was performed on the data that included only those respondents from Years 7 and 8. This subset of 834 respondents satisfied the criteria for a well-fitting model. The partial chi-square value of 416.16, with 9 degrees of freedom for the contingency table, confirmed the existence of a significant association between the variables ($p<0.0005$). The familiar saddle shape was once again in evidence among this older group of children. Thus while the relative frequencies of the cells changed to reflect the older children's responses, the nature of the association between the variables did not change, inspite of the early algebraic instruction the Year 7 and 8 children had received.

Discussion

The ability to express generality does not appear automatically in all children, nor is it an intrinsic skill that can be called on at will. Further the use of algebraic symbolism seems to be only associated with one category of natural language descriptions. While the various curriculum documents referred to at the beginning of this paper supported the focus on language development, they did not make explicit the precise nature of the language required for symbolic notation to become a natural extension of children's communication strategies.

The above results established a statistical association between natural language and symbolic language. The issue that arises from the identification of this association is whether the functional language description is a necessary precursor for the successful use of algebraic notation. (The data did not provide evidence of any cause and effect relationship between the components since a temporal difference in the appearance of the response sets was beyond the scope of the survey.) More specifically, can some children use sophisticated symbolic language without using natural language of a similar level of sophistication? Within the context of this study the issue is: can children provide accurate algebraic descriptions of number patterns without functional relationships in natural language?

Across the four stimulus items approximately 80% of all algebraic responses were associated with a functional description in the natural language component. What of the other 20%? Had this group of children been forced to reconceptualise the pattern when asked to provide a symbolic response? Were they able to produce a symbolic response independently of their natural language? Is the association between natural language and algebraic symbolism a transitory one? It could be that the process is analogous to a person using a foreign language. Initially the person formulates a response to some stimulus in his or her native tongue and then translates it into the new language. As competence in the new language develops people formulate responses in the new language thus bypassing the native language. It seems plausible to suggest that beginning algebra students need to formulate descriptions of patterns and relationships in their natural language prior to translation into the symbolic language of algebra. As confidence and competence develop, the natural language can be bypassed and descriptions formulated directly into the symbolic language. The opportunity to pursue these and other issues was afforded by the more detailed investigation of the student interviews conducted in a longitudinal study. Space restricts our ability to discuss these issues further in this paper, however, they are discussed elsewhere (Redden 1995b).

Indicative of the findings of the longitudinal study, was the comment by Erica, who was a subject interviewed over a two year period. During this time her pattern descriptions had become more sophisticated, while her use of symbolic notation had remained strictly arithmetic in nature and intent. Finally, after struggling to produce a functional description of a number pattern involving both a dependent and independent variable she began the task of producing a symbolic description of her deliberations. After some consideration, and a perplexed frown upon her brow she offered the observation:

> Wouldn't it be good if we had a symbol to stand for any number.

References

Arzarello, F. (1991). Procedural and relational aspects of algebraic thinking. In F. Furinghetti (Ed.), *PME 15*, 1 (pp. 80 - 87). Assisi: International Group for the Psychology of Mathematics Education.

Australian Education Council (1990). *A National Statement on Mathematics for Australian Schools*. Carlton: Curriculum Corporation.

Board of Secondary Education, New South Wales. (1988). *Mathematics Syllabus Years 7-8*. Sydney: Board of Secondary Education.

Ericksen, D. E. (1988) *Students' Conceptions of Variables and their uses for Generalisation of Mathematical Patterns*. Ph. D. thesis, Michigan State University.

MacGregor, M., & Stacey, K. (1993). Seeing a pattern and writing a rule. In I. Hirabayashi (Ed.), *PME XVII*, 1 (pp. 181-188). Tsukuba: Program Committee of the 17th PME Conference.

Mason, J., Graham, A., Pimm, D. & Gowar, N. (1985). *Routes to Roots of Algebra*. Milton Keynes: Open University Press.

Meira, L. L. (1990). Developing knowledge of functions through manipulation of a physical device. In G. Booker, P. Cobb, & T. N. de Mendicuti (Eds.), *PME 14*, 2 (pp. 101-108). Mexico: Program Committee of the 14th PME Conference.

Norusis, M. J. (1990). *SPSS Advanced Statistics User's Guide*. Chicago: SPSS Inc.

O'Brien, T. C. (1991). The Status of children's construction of relationships. In F. Furinghetti (Ed.), *PME15*, (pp. 117-120). Assisi: International Group for the Psychology of Mathematics Education.

Pegg, J. E., & Redden, E. (1990a). Procedures for, and experiences in, introducing algebra in New South Wales. *Mathematics Teacher*, 84(5), 386-391.

Pegg, J. E., & Redden, E. (1990b). From number patterns to algebra: The missing link. *The Australian Mathematics Teacher*, 46(2), 19-22.

Pegg, J. E. & Redden, E. (1990c). Student Experiences in developing algebraic symbolismusing number patterns. Paper presented to 13th annual conference of the Mathematical Research Group of Australasia. Hobart, Australia.

Redden, E. (1993)The transition from arithmetic to algebraic thinking. In Sutherland, R. (ed) *PME Working Group: Algebraic Processes and Structure*. PME XVII Tsukuba. Japan.

Redden, E. (1994) Providing an overview of qualitative data sets: Log-Linear modelling. MERGA 17. Lismore. Mathematics Education Research Group of Australasia, 523-530.

Redden, E. (1995a) *A longitudinal Investigation into Children's Understanding of Number Patterns and the Consequent Emergence of Algebraic Concepts*. Unpublished Doctoral Dissertation. Armidale. University of New England.

Redden, E. (1995b) MERGA 19 (in press). Melbourne. Mathematics Education Research Group of Australasia.

Rojano, T. & Sutherland, R. (1993). Towards an algebraic approach: The role of spreadsheets. In I. Hirabayashi (Ed.), *PME XVII*, 1 (pp. 189-196). Tsukuba: Program Committee of the 17th PME Conference.

Romberg, T. A. (Ed.). (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston: National Council of Teachers of Mathematics.

Ursini, S. (1991). First steps in generalisation processes in algebra. In F. Furinghetti (Ed.), *P.M.E. 15*, (pp. 316-323). Assisi: International Group for the Psychology of Mathematics Education.

Vygotsky, L. S. (1986). *Thought and Language*. Cambridge, MA: MIT Press.

ENACTIVISM AS A METHODOLOGY

David A. Reid
Memorial University of Newfoundland

Abstract *As research is learning, theories for learning and research methodologies in mathematics education overlap. For the Enactivist Research Group, enactivism is both the theoretical framework and the methodology for our research. Key ideas such as autopoesis, structure determinism, structural coupling, and coemergence are used to make sense of the learning of all participants in research, researchers included. This paper describes these key ideas and enactivist research methodology in mathematics education.*

Introduction

In his plenary paper at PME-18 John Mason (1994) reminded us of the interconnection between the theories for learning we employ as psychologists of mathematics education, and the methodologies we employ as researchers. It is obvious that this must be the case when we consider that our research is a particular instance of human learning, and ought to be understood in the same conceptual frame as that which we use to understand human learning of mathematics. In the following I describe how the theories of Maturana, Varela, Lakoff, and Johnson (among others) have informed and defined the research methodology of the Enactivist Research Group, as a model for enactivist research in mathematics education.

Enactivism, and Experientialism and Embodied Cognition.

While there have been some recent expositions on enactivist theory at PME (Edwards & Núñez, 1995) and in journals (Davis, 1995) a brief review is in order. Such a review of an entire theoretical perspective is in many ways a futile endeavor, but I hope here to touch on those points which are important to the following discussion of methodology, and to make some connections and contrasts with radical and social constructivisms.

Many of the ideas of enactivism can be found in the works of Merleau-Ponty (1962), Wittgenstein (1958), and Bateson (1987), but the first presentation of these ideas as a general theory for cognition comes in the works of Maturana and Varela (Maturana, 1987; Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991). They describe and name the key concepts of *autopoesis*, *structure determinism*, *structural coupling*, and *coemergence*. These ideas complement the experientialism of Lakoff and Johnson to produce a theory for cognition as "the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs" (Varela, Thompson & Rosch, 1991, p. 9).

The idea of autopoesis will seem familiar to constructivists, especially of the more radical variety. Autopoesis refers to that property of complex dynamic systems of spontaneous self-organization. The components of autopoetic systems "must be dynamically related in a network of ongoing interactions" (Maturana & Varela, 1992, pp. 43-44). That is, the components interact in ways which are continually changing, but which at the same time allow for the continuation of interactions so that the system continues to exist. In addition, the interactions of the components of an autopoetic system are responsible for the production of the components themselves. Autopoetic entities come into existence as a result of their own properties, and also maintain their existence by modifying their own properties.

> The problem is how to handle the problem of structural change and to show how an organism, which exists in a medium and which operates adequately to its need, can undergo a continuous structural change such that it goes on acting adequately in its medium, even though the medium is changing. Many names could be given to this; it could be called learning. (Maturana, 1987, pp. 74-75)

Learning then, for enactivists, is precisely this continual change which allows the learner to continue to function as an individual in a medium. Some social constructivists will be pleased to know that *any* sufficiently complex dynamic system can be described as autopoetic, and an enactivist description of an individual's learning could be applied just a well to a community as a whole.

Another idea which will be familiar to constructivists is that of structure determinism. What an autopoetic entity does is determined by its own structure, not by an external stimulus, which might trigger some action the structure was determined to do. There is an important distinction to be made, however, with some constructivist perspectives. It is not a matter of an individual having a cognitive structure, which determines how the individual can think, or of there being conceptual structures which determine what new concepts can develop. The organism as a whole *is* its continually changing structure which determines its own actions on itself and its world. This holistic vision of the cognitive entity is central to the idea of embodied cognition, described by Lakoff (1987), Johnson (1987), and Edwards & Núñez (1995).

It is possible, in fact probable according to empirical observation and complexity theory (Kauffman, 1993, 1995), that autopoetic entities organize themselves into networks of inter-action. When entities are in such a state, we say they are structurally coupled.

> If I have a living system ... then this living system is in a medium with which it interacts. Its dynamics of state result in interactions with the medium, and the dynamics of state within the medium result in interactions with the living system. What happens in interaction? Since this is a structure determined system ... the medium triggers a change of state in the system, and the system triggers a change of state in the medium. What change of state? One of those which is permitted by the structure of the system. (Maturana, 1978, p. 75)

Each acts according to its structures, but those structures are such that actions become coordinated. From an evolutionary point of view this can be explained by claiming that organisms which structurally couple are more likely to survive, so such structures become the most common. Complexity theorists would argue that such structures are inevitable at certain levels of complexity. In any case, structural coupling is a integral part of learning, that is the self-modification of autopoetic entities.

Structural coupling tends to be self-reinforcing, either because of the structures on the entities involved, or because they form part of an autopoetic entity whose autopoesis requires maintaining structural coupling among its parts. A favorite example is the herd of antelope, which leaves a single member behind when it moves from hilltop to hilltop, to act as a sentry for the herd as a whole. This particular interaction between that antelope and the herd threatens that individual, but for the herd as a whole it is a form of structural coupling which allows the continuing existence of the herd as an autopoetic entity.

Enactivist Methodology

Enactivism, as a methodology, a theory for learning about learning, addresses several levels of the activity of research. The level most familiar to most of us will be the interrelationship between researcher and data, in which we find ourselves learning new things within a context which is partially of our own creation. Enactivism can also be used to talk about the interrelationships in the research community, in which we as autopoetic researchers engage with other researchers in ways which preserve the structural coupling between us. A third level is that of coemergent autopoetic ideas which live in the medium of our minds, and of which we are emergent phenomena (as the herd is of the antelope).

A stereotypical image of research is the "experiment" in which we create a controlled situation, set events in motion, and impartially observe the results. This stereotype has already been extensively critiqued in the philosophy of science, so I will restrict myself to describing the enactivist alternative. In all research we establish a relationship, a structural coupling, with the milieu which is to be our topic of study. We interact with the people, objects, chemicals, and ideas we find there. By so doing we modify the milieu for each of its inhabitants, and the autopoetic entities in the milieu adapt in ways determined by their structures but triggered by our presence. At the same time we engage in the process we are there for, adapting in response to the triggers offered by the milieu. Note that an important part of our structures are our theories, beliefs, and biases. The changes which can be triggered in us, that is, what we can learn about the research context, are determined by our theories, beliefs and biases. What we learn is determined by what we know.

When I refer to the people, objects, chemicals, and ideas in a research situation, I refer to what is usually considered to be the source of "data" which is then interpreted. And there is "data" in enactivist research. The data generated in my research include field notes, video tapes, audio tapes, participants' writings, transcripts, notes based on viewing video tapes, mathematical activity traces which summarize the actions in a video taped session, research reports, conference presentations, and notes from discussions with other researchers. These artifacts can be lumped together as "data", but at the same time all of them record acts of interpretation, or a researcher learning in coemergence with a research situation. It can be said that there is no data, only interpretations and interpretations of interpretations. This is an important point to keep in mind, although I will be using "data" and "interpretation" interchangeably in the following, mostly to improve readability.

As a community, researchers in a field form the context in which their research occurs. I have to learn in ways which allow me to remain in interrelation with the participants and other aspects of my own research, and simultaneously in ways which allow me to remain a member of this research community. This establishes constraints. There are constraints offered by research data and constraints offered by the research community. I cannot know that the students I have worked with learn mathematics by a undertaking a series of gymnastic maneuvers and remain in interrelation with those students, or my video tapes of them. The "data" forbids some hypotheses. At the same time I cannot attribute their learning to messages beamed into their minds from outer space, even thought the data offers nothing to disprove this possibility, because such a hypothesis would sever my structural coupling with my research community.

The analysis of data in enactivist research can also be seen as a process of co-evolution of ideas. Theory and data coemerge in the medium of the researcher. The necessity of theory to account for data results in a dialogue between theory and data, with each one affecting the other. As enactivist researchers we attempt to make use of this interaction to transform the analysis of data into a continual process of change and encourage this process as the mechanism of our own continuing learning.

An Example of Enactivist Research

Over the past three years the Enactivist Research Group has been engaged in exploring enactivism both as a theory for learning and as a research methodology (Kieren, Gordon Calvert, Reid & Simmt, 1995; Gordon Calvert, Kieren, Reid & Simmt, 1995; Reid 1995). In doing so we have explored two key features of enactivist research, the importance of working from and with multiple perspectives, and the creation of models and theories which are good-enough *for*, not definitively *of*.

Multiple perspectives can refer to many aspects of enactivist research. The most obvious is the participation of a number of researchers, each with her or his own agenda, theories, and background. Enactivist research differs from collaborative research in that there is no common goal or question in which we are all interested (beyond the general nature of cognition). Particular research interests of the group include deductive reasoning, conversation, recursive models of understanding, and mathematical beliefs. At the same time we work with a common collection of data, about which we each reach conclusions related to our own interests and theories. These conclusions need not be parts of a single consistent whole. In fact, particular interpretations might be quite different. While some interpretations are not accepted by every member of the group, all interpretations must be explicable. That is it must be possible to explain the conceptual structure in which the interpretation holds, even to others who may not see the world thorough that conceptual structure. I use the phrase "multiple consensual contradictory perspectives," where "consensus" is used to mean the explanation of interpretations in way which make sense to others, to capture the important features of perspectives in enactivist research.

There are other ways in which multiple perspectives emerge. One is through multiple revisitations of data which brings a researcher to a situation with new theories and aims which represent the current structure of an ever changing being. Another is through the examination of a wide range of data. The aim here is not to come to some sort of "average" interpretation that somehow captures the common essence of disparate situations, but rather to see the sense in the range of occurrences, and the sphere of possibilities involved. A third source of perspectives is the act of communicating our research to others. By so doing we invite audiences and readers to engage with us in enactivist research producing their own interpretations of our ideas and data.

The selection of this wide range of data is not always a matter of planning. Part of working with the Enactivist Research Group is continually encountering new situations which occasion reflection and interpretation. For each of us the data we see is in some ways "found" and made sense of. This aspect of enactivist research has been called "bricological" (Reid, 1995). Bricological research combines the flexibility and creativity of *bricolage*, with an underlying *logic* of inquiry. Bricolage, as it is used in conceptualizing bricological research, favors the production of complex structures, theories, models, etc. appropriate to research on complex systems such as human learners and societies. It can be contrasted with a technological attitude that favors production of lots of results through straightforward, "clean" techniques. The logic of the bricological methodology comes from the questions chosen for research, and the theories and models with which the research begins. These questions, models, and theories reflect expectations of what might be seen. These expectations correspond to the plastic structure that determines the actions of an individual in a context. Just as an individual's structure changes in changing the context, so our expectations change even as we observe, interview, and analyze according to our expectations.

It is important to note that the theories and models of enactivist research are not models *of*. That is to say they do not purport to be representations of an existing reality. Rather they are theories *for*; they have a purpose, clarifying our understanding of the learning of mathematics for example, and it is their usefulness in terms of that purpose which determines their value. The recursive dynamical models for understanding, developed by Kieren and Pirie (Kieren, Pirie & Reid, 1994) and the language for discussing reasoning developed by Reid (1995) are two examples of theories *for*.

Conclusion

As researchers we search for understanding of the learning of mathematics, making use of psychological perspectives, theories for learning, to make sense of what we see. I have given an example here of how one such perspective acts also to make sense of what we do. Enactivism, based on an equation of knowing, being and doing, provides a context in which it is easy to see research about learning as a form of learning. It is not special in this regard, but I hope that by presenting enactivism as a methodology I have presented it as a theory for learning better than I might have, and that by describing that methodology through a theory for learning I have been able to communicate the spirit of the research that is done by the Enactivist Research Group.

References

Bateson, G. (1987). *Steps to an ecology of the mind.* North Vale NJ: Jason Aronson.

Davis, B. (1995). Why teach mathematics? Mathematics education and enactivist theory. *For the Learning of Mathematics, 15*(2), pp. 2-9.

Edwards, L. & Núñez, R. (1995). Cognitive science and mathematics education: A non-objectivist view. In *Proceedings of the Nineteenth Annual Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 240-247). Recife.

Gordon-Calvert, L., Kieren, T., Reid, D. & Simmt, E. (1995). *Thinking otherwise: Enactivism, mathematics and education.* Paper presented at the meeting of the

Canadian Society for the Study of Education, Montreal.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason*. Chicago: University of Chicago Press.

Kauffman, S. (1993). *The origins of order: Self-organization and selection in evolution.* Oxford: Oxford University Press.

Kauffman, S. (1995). *At home in the universe: The search for laws of self-organization and complexity.* Oxford: Oxford University Press.

Kieren, T., Pirie, S. & Reid, D. (1994). Mathematical Understanding: Always under construction. In J. da Ponte & J. Matos, (Eds.), *Proceedings of the Eighteenth Annual Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 49-56). Lisbon.

Kieren, T., Gordon-Calvert L., Reid, D. & Simmt, E. (1995). *An enactivist research approach to mathematical activity: Understanding, reasoning, and beliefs.* Paper presented at the meeting of the American Educational Research Association, San Francisco.

Lakoff, G. (1987). *Women, fire and dangerous things: What categories reveal about the mind.* Chicago: University of Chicago Press.

Mason, J. (1994). Researching from the inside in mathematics education: Locating an I-you relationship. In J. da Ponte & J. Matos, (Eds.), *Proceedings of the Eighteenth Annual Conference of the International Group for the Psychology of Mathematics Education*, (Vol. I, pp. 176-194). Lisbon.

Maturana, H. & Varela, F. (1992). *The Tree of Knowledge: The biological roots of human understanding.* Boston: Shambhala.

Maturana, H. (1987). Everything said is said by an observer. In W. Thompson (Ed.), *Gaia: A way of knowing.* (pp. 65-82). Hudson, NY: Lindisfarne Press.

Merleau-Ponty, M. (1962) *Phenomenology of perception.* Trans. Colin Smith. London: Routledge & Kegan Paul.

Reid, D. (1995). *The need to prove*. Unpublished doctoral dissertation, University of Alberta, Department of Secondary Education.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge MA: MIT Press.

Wittgenstein, L. (1958). *Philosophical investigations.* (G. E. M. Anscombe, Trans.). Oxford: Basil Blackwell.

TEACHING LINEAR ALGEBRA : ROLE AND NATURE OF KNOWLEDGES IN LOGIC AND SET THEORY WHICH DEAL WITH SOME LINEAR PROBLEMS

Marc Rogalski (Didirem, Paris-Lille)

Abstract. *We show by empirical studies of student's products that solving linear algebra problems which deal with inclusions or intersections of subspaces requires particular abilities in logic and set theory. We precise the specific nature of this abilities, which is linked to the logical and set-theorical status of the notions of equations and parametric representations for a geometrical object.*

Résumé. *Nous mettons en évidence, à travers l'analyse de productions d'étudiants, que la résolution de problèmes d'algèbre linéaire où interviennent inclusions et intersections de sous-espaces vectoriels exige des compétences particulières en logique et théorie des ensembles. Nous précisons la nature spécifique de ces compétences, qui concernent le statut logique et ensembliste des notions d'équations et de paramétrage d'un objet géométrique.*

Several studies have dealt with teaching and learning linear algebra at the beginning of university courses. For example, in anglo-saxon context, some references are Harel (1989), Hillel and Sierpinska (1994) ; in the french context references are Artigue and Dias (1995) and Dorier, Robert, Robinet and Rogalski (1994) ; see also Rogalski (1995). A recent book edited by Dorier (1996) proposes a synthesis of problems encountered in teaching linear algebra. In the french context, one of the three mains ideas developped is that there are prerequisites settings, in particular logic and set theory (LTS). The present paper explains the result of empirical studies, led in collaboration with A. Robert and J. Robinet, in order to precise this hypothesis, which was preciously expressed in Robert and Robinet (1989) and comforted by Dorier's work (1990).

The hypothesis of LST prerequisites is studied through a detailed analysis of student's performance and activity when solving problems dealing with subspaces inclusion or intersection. Two versions of a same problem : "the hyperplanes problem" (HP) were used ; the first one - called "the 1992 examination" - was proposed in 1992 to 125 students ; the second one - called "the 1994 workshop" - was solved by small groups (4 students) during training sessions in another university year (1994). We will first present the two situations. Then student's procedures and failures due to LTS, in 1992 examination, and detailed verbal interaction during problem solving in the 1994 workshop will successively be analysed. Overall conclusions will be drawn concerning LST prerequisites, and finally we presents further hypothesis and suggestions for teaching.

I. The hyperplanes problem

I.1 The 1992 examination

The original text of the hyperplanes problem (HP)

" Let E_1, E_2 and E_3 be subspaces of $\mathbb{R}^4$ defined by equations e_1, e_2 and e_3 :

$$e_1 : 3x - 2y - z + t = 0, \quad e_2 : x + y + 2z - t = 0, \quad e_3 : 5x + 3z - t = 0.$$

(a) Compare E_3 and $E_1 \cap E_2$. (b) Determine all subspaces of $\mathbb{R}^4$ containing $E_1 \cap E_2$, and precise how you choose to search and represented them. "

The HP was part of a partial examination involving also three other exercices (about calculus). It was given at the middle of the university year. Previous teaching in linear algebra was : Gauss method for linear equations, geometry in $\mathbb{R}^3$ including pencil of planes, theory of rank of vectors in $\mathbb{R}^n$, and dimension of subspaces in $\mathbb{R}^n$, with the twofold definitions, through linear equations or by linear parametrical representation.

In this paper, our results will concern question (a) only.

I.2 The 1994 workshop

In a "workshop", there are seven groups of four students in the class-room ; each group has one hour and a half for solving a problem without indication and write his solution. The teacher interacts with a group only two or three times. Verbal interactions give us a better view of difficulties for solving the problem.

The new text of the hyperplanes problem

" 1°/ Two subspaces E_1 and E_2 of $\mathbb{R}^3$ are given by equations e_1 and e_2 :

$e_1 : 3x - 2y - z = 0$; $e_2 : 2x - y + 2z = 0$.

Determine all subspaces of $\mathbb{R}^3$ containing $E_1 \cap E_2$.

2°/ Three subspaces E_1, E_2 and E_3 of $\mathbb{R}^4$ are given by equations e_1, e_2 and e_3 :

$e_1 : 2x + y - z + t = 0$; $e_2 : x - y + 2z - t = 0$; $e_3 : 4x - y + 3z - t = 0$.

(a) Compare E_3 and $E_1 \cap E_2$. (b) Determine all subspaces of $\mathbb{R}^4$ containing $E_1 \cap E_2$, while specifying how you have chosen to represent them. "

Changes with respect to "1992 examination"

- The workshop hapened at the same place in the university year, and more time was given for solving the problem.
- Changes were introduced in the text - begining by the question of the subspaces containing $E_1 \cap E_2$, and in $\mathbb{R}^3$ instead of $\mathbb{R}^4$: students have be taught about pencils of planes in $\mathbb{R}^3$, and we hope to enable - through analogy - a transfer to $\mathbb{R}^4$.
- In 2°/ (a), students are asked for giving two methods in order to impulse interrogations about methods in 2°/ (b).

In this paper results will only be concerned with the impact of insufficient knowledge in LTS. Other aspects of the HP are studied in the book edited by Dorier (1996).

II. Strategies, procedures and LTS prerequisites in the "1992 examination"

II.1 Mathematical analysis of the hyperplanes problem

For the 1992 examination, analysis is given for the question (a) only. Possible strategies (which can be correct or false in their realization) dealing with objects and notions of linear algebra are presented in figure 1. Abrevations are explained below the figure, and detailed solution ig given then.

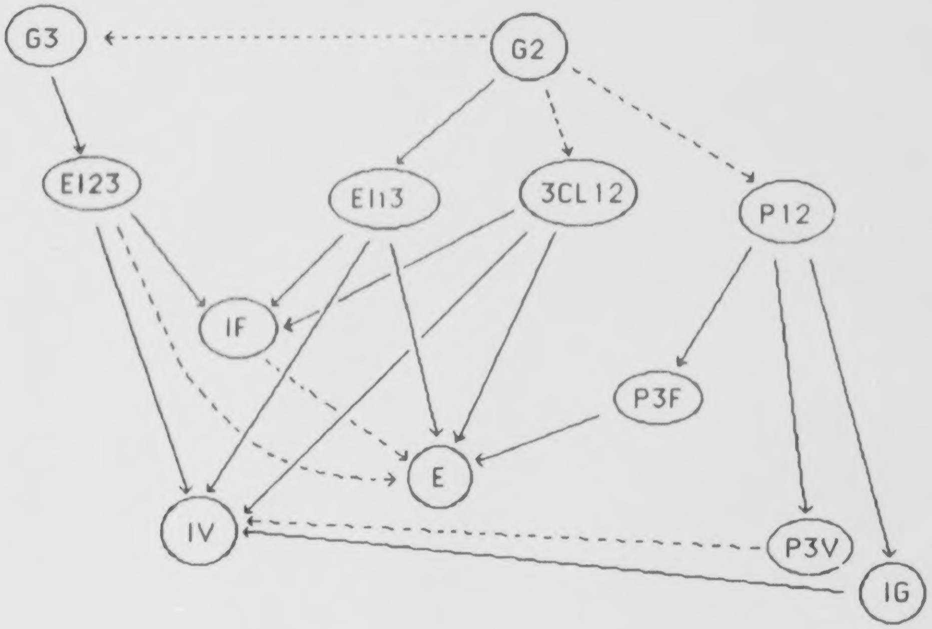


Figure 1 : graph of procedures

Dotted arrows mean that the deduction may be implicite in students copies.

Legend for the procedures

G3 : applies Gauss method to e_1, e_2 and e_3

G2 : applies Gauss method to e_1 and e_2

IG : looks if generators of $E_1 \cap E_2 \in$ or $\notin$ E_3

EI23 : proves that $E_1 \cap E_2 \cap E_3 = E_1 \cap E_2$

P3F : gives a false parametric representation for E_3

P12 : gives a parametric representation of $E_1 \cap E_2$

IF : says the (false) inclusion $E_1 \cap E_2 \supset E_3$

IV : says the (true) inclusion $E_3 \supset E_1 \cap E_2$

E : says the (false) equality $E_3 = E_1 \cap E_2$

EIi3 : proves that $E_1 \cap E_2 \cap E_3 = E_i \cap E_3$ $(i = 1, 2)$

3CL12 : says that $e_3 = \alpha e_1 + \beta e_2$

P3V : gives a correct parametric representation for E_3

The detailed solution

The Gauss method on equations can be writen under two different forms :

$$\boxed{G2} \to \begin{array}{c} -t + 2z + x + y = 0 \\ \vdots \\ z + 4x - y = 0 \end{array} \quad \text{or} \quad \boxed{G3} \to \begin{array}{c} -t + 2z + x + y = 0 \\ z + 4x - y = 0 \\ 0 = 0 \end{array}$$

for between equations there is the relation $e_3 = e_1 + 2e_2$: $\boxed{3CL12}$

* Hence $E_1 \cap E_2 \cap E_3 = E_1 \cap E_2$: $\boxed{EI23}$. With a variant G'3 of G3, e_i and e_3 (i = 1 or 2) remain : it is $\boxed{EIi3}$, hence $E_1 \cap E_2 = E_i \cap E_3$. From there, one can guess an inclusion or an equality between $E_1 \cap E_2$ and E_3. If one has understood the elementary set theory, one conclude $\boxed{IV}$, if not one may conclude $\boxed{IF}$ or $\boxed{E}$.

* From G2, one can give a parametric representation of $E_1 \cap E_2$ by choosing $x = u$ and $y = v$; this gives $\boxed{P12}$: $x = u$, $y = v$, $z = -4u + v$, $t = -7u + 3v$; that is $X = uU + vV$, where $U = (1, 0, -4, 7)$ and $V = (0, 1, 1, 3)$ are generators of $E_1 \cap E_2$. Then one immédiately verifies that U and V satisfy equation e_3 : it is $\boxed{IG}$. Hence one gets inclusion $E_1 \cap E_2 \subset E_3$.

* Equation $e_3 : 5x + 3z - t = 0$ gives a parametric representation for E_3 :

$x = \lambda$, $y = \mu$, $z = \nu$, $t = 5\lambda + 3\nu$; it is $\boxed{P3V}$, or $\boxed{P3F}$ in case of error in calculation. For identifying the intersection of $E_1 \cap E_2$ with E_3, one may search condition on (u, v) for the existence of (λ, μ, ν) satisfying : $u = \lambda$, $v = \mu$, $-4u + v = \nu$, $-7u + 3v = 5\lambda + 3\nu$. For every (u, v), one can compute λ and μ ($\lambda = u$, $\mu = v$), and ν may be calculated in two ways : $\nu = -4u + v$, and $3\nu = -7u + 3v - 5\lambda = -7u + 3v - 5u = -12u + 3v = 3(-4u + v)$: one obtains the same value. Hence $\forall(u,v)$, that is $\forall m \in E_1 \cap E_2$, there exist (λ, μ, ν) such as …, that is $m \in E_3$. Hence one gets the inclusion $E_1 \cap E_2 \subset E_3$.

II.2 Students productions

The overall performance is : 30 success, 88 failures, 7 "problem not processed".

Concerning objects and notions of linear algebra usefull for solving the problem, a distinction may be done between students showing deep misunderstanding and confusions, and students showing an operational knowledge and using them for built a "reasonable strategy" which may lead to success. This strategies are in figure 2.

Only 37,5% of the reasonable strategies lead to success.The main causes for failure concern 61% of students with these strategies, and are the following :

writing inclusion in the wrong direction (IF) ; writing equality instead of inclusion (E) ; difficulties in comparing the parametric representations of the subspaces.

These causes of failure deal with notions belonging to LST :

adding conditions when defining a set means that it becomes larger or smaller ? what really differs between inclusion and equality of sets ? what can be deduce from the set equality $A \cap B = B$? how to prove an inclusion (let $x \in A$, then ..., hence $x \in B$) ? how do quantificators play in comparison of two parametrized sets ? why is it necessary to give different names to the parameters ?

Strategy 1	{G2 or/and G3 → 3CL12 or EI23 or EIi3 → IV or IF or E}	63	(50%)
Strategy 2	{P12 → P3F or P3V → Comparison of parametric repres.}	11	(9%)
Strategy 3	{P12 → IG}	6	(5%)
Total of "reasonable strategies"		80	(64%)

Figure 2 : distribution of reasonable strategies

In figure 3, we present the distribution of LST errors. The percent of "good" students, that is implementing a reasonable strategy, but without success because of LST weakness (37%, 41+8 students) is of the same order of magnitude that of "bad" students, unable to implement any coherent strategy.

Moreover, the importance of LST errors is a non decreasing fonction of the "quantity" of LST dealing in the choosen strategy : 0% in the strategy 3 (when the generators are finded, it is easy to see if they belong to E3), 65% in the strategy 1 (which use comparison of sets and number of conditions for defining a set) and 73% in the strategy 2 (more difficult for LTS because it use also quantificators).

These data show the importance of deficiencies in LST setting in difficulties concerning linear algebra, at least for problems of the type of the HP. This confirms Dorier's conclusions (1990). For identifying more precisely LST knowledges involved in solving such linear problems, we used verbal communications exchanged in group problem solving in the 1994 workshop.

Figure 3 : success, LST and other errors

production	performance	number	%
no strategy, no answer, notional confusions...	failures	45	36
strategy 1	success	22	50
	LST errors	41	
strategy 2	success	3	9
	LST errors	8	
strategy 3	success	5	5
	other errors than LST	1	
reasonable strategies		80	64

III. Semantics of LST errors in the HP : the 1994 workshop

Activity of three groups working on HP was recorded. We analysis mathematical difficulties of students in order to better understand their LST problems.

III.1 An example of chronology of work for one group

Question 1	8 mn	determining subspace $E_1 \cap E_2$, but forgoting the goal : determine subspaces containing $E_1 \cap E_2$.
Question 2(a)	10 mn	using Gauss method and parametrics for $E_1 \cap E_2$.
	25 mn	trying, without success, to interpret the result as an inclusion, with 2 mn of teacher's support ; the "false" inclusion seems them intuitive, but is in contradiction with the dimension...
Question 2(b)	18 mn	unsuccess and discouragement, in spite of 3 mn of teacher's support ; a third intervention of 1 mn releases them : EUREKA !
	2 mn	rapid assesment and conclusion on all points (with a return to question 1).
Rédaction	12 mn	perfect writing of the solution ; netherless it is not sure that they are convinced by the reasons leading to the "good" inclusion $E_1 \cap E_2 \subset E_3$.
Total :	1 hour 15 mn.	

In this example, the group try to use strategy 2, with LST difficulties. Netherless success is triggerd by short teacher's interventions. We will detail two main points in the LST difficulties encountered in one or more groups.

III.2 Mathematical difficulties linked to weaknesses in LST knowledges

(a) *LST and bunches of planes*

* There is first a difficulty with the role of parameters s and t in the equations of planes of a bunch $sp + tq = 0$: students do not master the status of mute variables when they parametrize a family of *subsets* (this difficulty does not appear in dealing with linear combinations $su + tv$ of *vectors*).

* There is confusion between a bunch and the subspaces elements of the bunch. Quotations : "*You say that a bunch of planes - that is a vectoriel subspace of the straight line D ?*" [does he main : containing D ?] ... or : "... *bunches of planes are the only subspaces that* ...". Dialog about E_3 belonging to the bunch defined by E_1 and E_2 : "*E_3 contains $E_1 \cap E_2$*" - "*No, I think that it is E_3 that is included...*" ; "*E_3 is included in the hyperplanes bunch defined by E_1 and E_2*".

This difficulties in the LST setting are of the same type : il $\mathcal{F}$ is a bunch defined by a straight line D and E a subspace element of $\mathcal{F}$, what has to be written : $E \in \mathcal{F}$, or $E \subset \mathcal{F}$? $D \in \mathcal{F}$, or $D \subset \mathcal{F}$, or... ?

(b) *LST and the notion of equations of a geometric object*

* Students's exchanges reveal difficulties in correctly expressing relations between systems of equations and subspaces. The stable and convinced affirmation of the false inclusion $E_3 \subset E_1 \cap E_2$ seems to be due to a confusion equations/subspaces which

is classic : the linear combination $e_3 = ae_1 + be_2$ is automatically translated in this false inclusion.

* The main long term difficult in groups work depends of set theory in two ways :

- What can be concluded from the equality $E_1 \cap E_2 \cap E_3 = E_1 \cap E_2$? General set theory is poorly known by students and its previous functionning in concrete situations was been scarce. Hence, they are unable to control their conviction with set theory methods : to drawn a Venn diagram, or to make a standard reasonning on sets as : "let x be an element of A let us express what it means and let us deduce that x belongs to B...".

- Students have no possibility to make a control through considering the number of constraints (number of equations : adding constraints restraints sets) because for them "equations of a subspace" does not means "constraints on a point (x, y, z) expressing that it belongs to a given subset".

It is like if the mapping between subspaces and equations is dependant of operational domain only, not of conceptual domain : it is possible with only calculations to go from equations to a basis of the associated subspace, and to find equations for a subspace given in parametric form. For students, the contract requires to apply some algorithms, and the strongness of linear situations [and their didactical weakness !] is that "it works" even when one does not understand what these algorithms mean. This could explain the confusion of the first point : student have not understand that equations and subspaces are not in the same domain, and that their logical extension in terms of set theory are in opposite direction.

To conclude, it is possible to identify precisely the nature of some LST difficulties which take place in linear algebra : from the point of view of logic and set theory, the status of the equations of a geometrical object as constraints expressing that a point belong to this object is not acquired by students. This reinforce the hypothesis expressed in Dorier (1990) that LST prerequisites for linear algebra present specificities linked to the notions of this mathematical domain.

IV. Some hypotheses and suggestions

We make some hypotheses for explaining this LST difficulty at the begining of university. Then, suggestions are proposed for remedying these problems.

IV.1 Origin of the difficulties concerning equations of a geometrical oject

In french curriculum equations of straight lines in plane are introduced at the 9° grade (14-16 year old students). But in fact students learn only algorithms to pass from a line to his equation and vice versa ; many rules are used, with various pedagogical practices in order to reinforce them, in such a way that, at the final time, the LST meaning of the concept of equation of a line has vanished. Then, equations of planes and lines in space are studied in the same maner, and inclusion or intersection problems are not studied.

IV.2 Few use is done of set theory before introducing linear algebra

Almost always, this theory is not really taught, even partially, before the university studies, and only in aroundabout way in the first university year. When set theory is presented, it is rarely used in a significant way, and no relevant advantage is taken of it (for example by interpreting universal implication as inclusion). So, we think that one has to really learn the elementary and "naive" set theory, and it is possible to make it not formal (see Legrand 1990) and to link it to mathematical concepts.

IV.3 Developping cartesian geometry with research of geometrical loci

Many poeple think that cartesian geometry is usefull for giving mental images for liner algebra. But, overall, we think it can illuminate the nature of the link between equations and geometrical objects, by oppening a field of examples and motivations. But it is necessary to use situations without algorithms, i.e. situations which are not linear : curves and surfaces. It would be also necessary to ask problems involving inclusions, equalities, intersections of such objects. In particular, researching "non linear" geometrical loci, with changes between geometrical and analytical settings, and changing point of view between equations and parametric representations, could offer a way for supporting the acquisition of the LST status of the notion of equations of geometrical objects, which was shawn as a key-point in LST difficulties for students at the begining of linear algebra teaching.

References

ARTIGUE M., DIAS M., Articulation problems between different systems of symbolic representations in linear algebra, *Proceedings of PME XIX*, Lisbon.

DORIER J.L.,1990, *Contribution à l'étude de l'enseignement à l'université des premiers concepts d'algèbre linéaire. Approches historique et didactique*, Thèse de doctorat de l'Université J.Fourier-Grenoble 1.

DORIER J.L., ed., 1996, *L'enseignement de l'algèbre linéaire en question (panorama de la recherche didactique sur ce thème)*, La Pensée Sauvage, Grenoble.

DORIER J.L., ROBERT A., ROBINET J. et ROGALSKI M., 1994, The teaching of linear algebra in first year of french science university : epistemological difficulties, use of the "meta lever", long-term organization, *Proceedings of PME XVIII*, vol. IV, 137-144, Lisbon.

HAREL G., 1989, Learning and teaching linear algebra, *For the learning of mathematics 11*, 139-148.

HILLEL J., SIERPINSKA A., 1994, On one persistent mistake in linear algebra, *Proceedings of PME XVIII*, vol. III, 65-72, Lisbon.

LEGRAND M., 1990, "Circuit", ou les règles du débat mathématiques, dans: *Enseigner autrement les mathématiques en DEUG A première année*, brochure de la Commission Inter-IREM Université, 1990.

ROBERT A., ROBINET J., 1989, Quelques résultats sur l'enseignement de l'algèbre linéaire, *Cahier de didactique des Mathématiques* n°53 IREM de Paris 7.

ROGALSKI M., 1995, Que faire quand on veut enseigner un type de connaissances tel, que la dialectique outil-objet ne semble pas marcher, et qu'il n'y ait apparemment pas de situation fondamentale ? L'exemple de l'algèbre linéaire, *Actes du Séminaire de Didactique des Mathématiques de Grenoble*, Grenoble, Imag.

WAYS OF SOLVING ALGEBRA PROBLEMS: THE INFLUENCE OF SCHOOL CULTURE

Teresa Rojano[1], Rosamund Sutherland[2]
Sonia Ursini[1], Susan Molyneux[2] and Emanuel Jinich[1]
[1] Center for Research and Advanced Studies, IPN, Mexico
[2] School of Education, University of Bristol, UK

Abstract

This paper presents the results of a subset of a Mexican/British collaborative project aimed at investigating cultural influences of mathematical practices in science. Here we report our analysis of the relationship between students' approaches to solving algebra problems and their previous school algebra experiences. The results point out that whereas there may be some invariant obstacles to algebraic thinking which go across cultures, there are also many ways in which the school mathematics culture determines the ways in which students approach algebra problems.

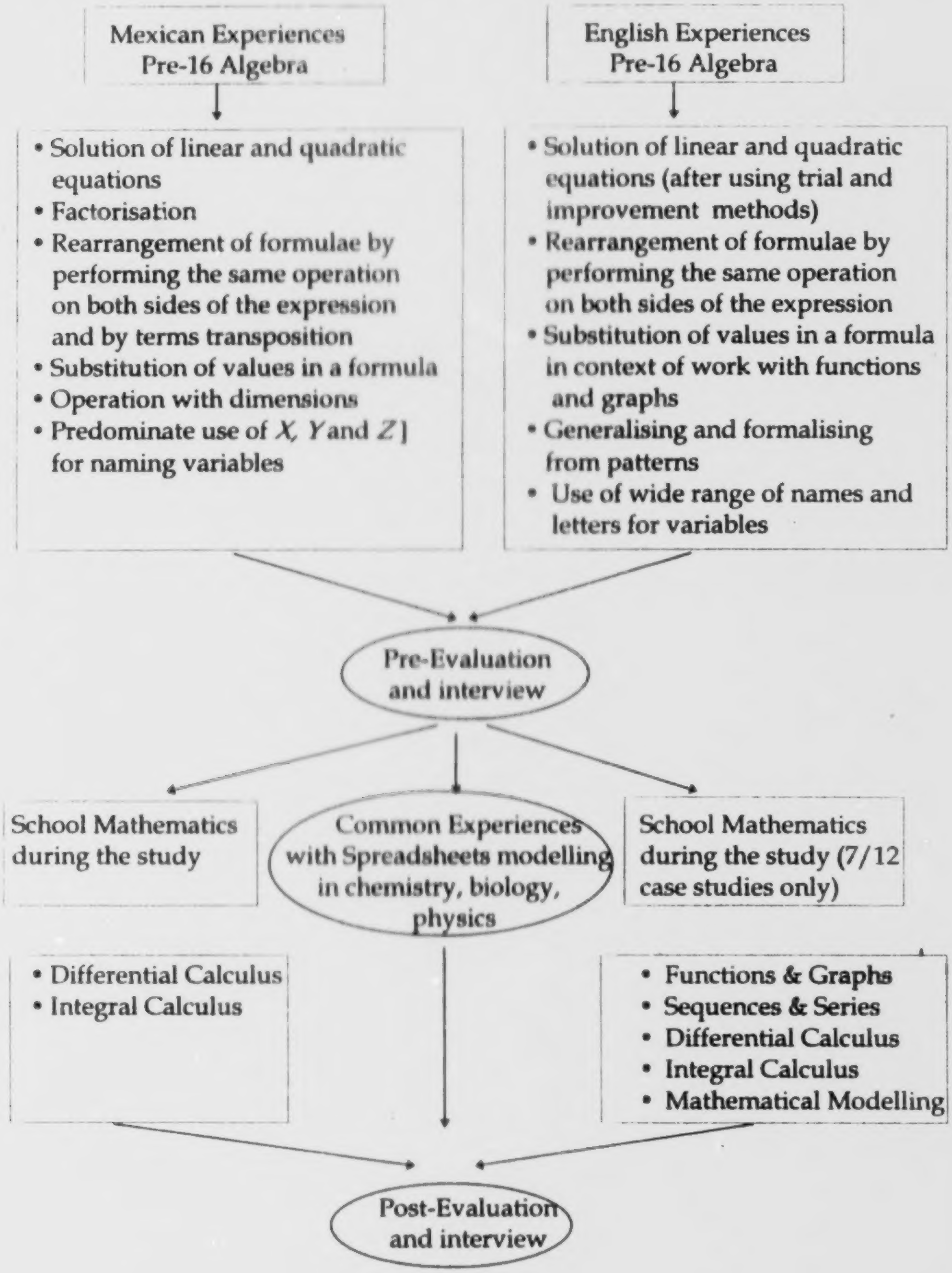
Introduction

Mathematics and more particularly algebra has been viewed as a universal generalising tool which can unproblematically be applied to different situations, and which always takes the same form in different settings. Research on children's difficulties with algebra has tended to assume that these difficulties are somehow independent of culture, school culture and types of problems being solved (e.g. Küchemann, 1980). Vygotsky's theory (1978) suggests that studies of pupils cannot be separated from influences such as school curriculum, school culture and the social milieu in which the teaching and learning is carried out. The work of Lave (1988) emphasises the way in which mathematics is used in different ways by students as a structuring resource for solving problems. The articulation of this structuring resource is more likely to be influenced by culture and problem situation than by a use of mathematics as a general organising resource.

In recently completed project we investigated the ways in which the previous mathematical experiences of students influenced the ways in which they tackled mathematical modelling problems in pre-university science courses. Within this project we worked with two groups of students, one in Mexico (9 students) and one in the UK (12 students). The general aim of the Mexican/British project was to investigate the mediating role of spreadsheets for expressing and solving mathematical modelling problems within biology, chemistry and physics (Sutherland et al., 1996). Students worked on a series of spreadsheet modelling activities within their science classes throughout one academic year. In this paper we report on the students' response to several algebra items (a subset of a more extensive science and mathematics evaluation) which were administered to the students at the beginning and end of the study. Individual interviews were also carried out with the students.

Students' Algebra Background

Table 1 presents the organisation of the Mexican/British project stressing pupils' different algebraic backgrounds, the common experience as well as the different teaching received in their math courses throughout the research project.


Mexican Experiences Pre-16 Algebra
English Experiences Pre-16 Algebra
• Solution of linear and quadratic equations
• Factorisation
• Rearrangement of formulae by performing the same operation on both sides of the expression and by terms transposition
• Substitution of values in a formula
• Operation with dimensions
• Predominate use of X, Y and Z] for naming variables
• Solution of linear and quadratic equations (after using trial and improvement methods)
• Rearrangement of formulae by performing the same operation on both sides of the expression
• Substitution of values in a formula in context of work with functions and graphs
• Generalising and formalising from patterns
• Use of wide range of names and letters for variables
Pre-Evaluation and interview
School Mathematics during the study
Common Experiences with Spreadsheets modelling in chemistry, biology, physics
School Mathematics during the study (7/12 case studies only)
• Differential Calculus
• Integral Calculus
• Functions & Graphs
• Sequences & Series
• Differential Calculus
• Integral Calculus
• Mathematical Modelling
Post-Evaluation and interview

Although in Mexico there has been some recent curriculum changes, the background of the subjects of the study corresponds to the old curriculum in which learning and use of rules and algorithms was emphasised. The teaching of algebra emphasised manipulation skills and algorithms for solving linear and quadratic equations. These topics are stressed as useful skills that students need to master in order to be able to work in university maths and physics courses, maths being taught in a decontextualized way. Moreover, in the pre-16 maths and physics courses the students were taught how to operate on literal symbols representing dimensions. It must be stressed that both textbooks and teachers in the classroom tend to use the symbols X, Y and Z for naming variables. Although other literal symbols are used, these are usually linked to some specific context as, for example, "v" for velocity. All 9 Mexican case study students followed a mathematics course during our study.

In the UK there have been quite considerable changes to the algebra curriculum over the last ten years, which have been partly influenced by the results of research on children's' difficulties with algebra (for example the CSMS study, Hart et al., 1981). In contrast to Mexico, symbol manipulation skills and algorithms for solving equations are no longer emphasised. The students in this study would have spent more time on ideas related to functions and graphs than on manipulative algebra. They had also been taught trial and refinement techniques for solving linear and quadratic equations (using a calculator) and were more likely to use this approach than an analytical algebraic method. The UK students of the project were studying three A-levels, predominantly chosen from biology, chemistry, physics and mathematics. Only 7 of the 12 case study students were studying A-level mathematics. The other 5 students were not studying any mathematics in their pre-university course.

In this paper we present the results of the UK and Mexican students' responses to a number of questions which were answered in the pre and post-evaluation. Our analysis is also based on their responses to individual interviews.

Three Problems in the Pre and Post-evaluation

Conversion Problem. The following question was used to probe students' ways of establishing and expressing relations; as well as their interpretation and use of variables.

> *Write an equation to a) convert hours H into seconds S and b) to convert seconds S into days D.*

In the pre-evaluation, the majority of Mexican students (5/9) were not able to produce a correct formulation of the problem; 1/9 produced a correct formula only for part a); 2/9 produced correct formulae for parts a) and b). Analysis of their written responses shows that 4 out of 9 students interpreted the given literals H, S and D as shorthand for hours, seconds and days, and used them as tools for understanding the problem and finding out how to convert time units. For example, this is the case of Laura Elena (see Figure 1) who used H, S and D in her attempts at making sense of the problem. For this she applied her knowledge about operating with dimensions, a subject she was taught in pre-16 physics courses. Only 2 students used the given literals H, S and D as variables in their formulae, while 3 students introduced X and Y for representing variables (see Figure 1, Liztli). The introduction of X and Y for naming variables is likely to be a consequence of their predominant use, both in the Mexican textbooks and by the teacher in the classroom. It also suggests that the given literals H, S and D had a strong physical referent for these students which might have been an obstacle to using them as variables. The students' use of X and Y seemed to help them (for example, Liztli) to identify and symbolise the variables of the problem and obtain a correct answer.

	Part a)	Part b)
Laura Elena	$H \times \frac{S}{H} = \text{seg}$	$S \times \frac{\text{min}}{S} \times \frac{H}{\text{min}} \times \frac{D}{H} = \text{días}$
Liztli	1 hr = 3600 s X = horas , y = segundos 3600X = y	

Figure 1

In contrast, the answers given to this item by UK students showed that only 2 out of 12 students interpreted H, S and D as shorthand for hours, seconds and days. The majority (10/12) of the UK students wrote formulae using the given literal symbols H, S and D as variables which is likely to relate to their previous experience of using a wide range of variable names in functional relationships. However, the formulae produced by the majority of UK students suggest a wrong interpretation of the question posed; 3/12 produced a correct formula for part a), and only 2/12 produced correct formulae for parts a) and b). Instead of converting hours to seconds and seconds to days they wrote how many seconds are in one hour and how many seconds are in one day showing in this way a tendency *to conserve* and not *to convert* quantities. These incorrect responses (see Figure 2, Meimei) are similar to the "students&professor" error (Clement, 1982) although in the present case the source of the error not necessarily can be attributed to an erroneous translation from natural to algebraic language.

The nature of the Mexican students' responses to this item changed substantially in the post evaluation. After analysing the problem using the given literals, as well as whole or shortened words, all the Mexican students used the given symbols H, S and D to express the relationship; 4/9 produced a correct formula for part a) and 4/9 produced a correct formula for parts a) and b). This new confidence in the use of literal symbols, other than X and Y, to represent variables could be a consequence of the spreadsheet experience in which pupils had been using different names and letters for naming variables.

	Part a)	Part b)
Meimei	H = 5 x 60 x 60	D = 5 x H x 24

Figure 2

By the post-evaluation the majority of UK students produced responses in a correct form for both parts of the question, 3/12 for part a) and 6/12 for parts a) and b), the number of correct formulae also increasing for the UK students. These results suggest a better understanding of the question. This might be a consequence of the spreadsheet experience where the data of a problem had to be rearranged in order to fit the characteristics of the spreadsheet and feedback concerning the correctness of the interpretation of the problem was provided.

Rearranging a Formula. The following question was used to probe students' strategies for rearranging a formula in order to find an inverse function.

The relation between temperature expressed in Fahrenheit and in Celsius is given by the formula $F = \frac{9}{5}C + 32$. *What is the formula for converting the temperature Fahrenheit to centigrade?*

The Mexican students' answers to this item suggest the influence of two methods, namely, the balancing of an equation (that is performing the same operation on both sides of an expression in order to make one of the variables the subject of the equation) and the transposition of terms (that is transposing terms from one to the other side of an equation in order to make one of the variables the subject of the equation). These students had experience of both, balancing expressions and transposing terms in order to rearrange formulae in their pre-16 algebra courses, about two years before this study. The method taught at school in Mexico is usually that of balancing expressions but, as teachers often notice that students tend to shift to using the transposition method (often taught by older peers or parents), this method is also taught. The use of the transposition method was used by 4 out of the 9 students and is clearly expressed by the record made by Liztli (Figure 3).

$$F - 32 = \frac{9}{5}C$$

$$\frac{F - 32}{C} = \frac{9}{5} \qquad C = \frac{(F-32)5}{9}$$

Figure 3

However, a mixed approach, using both methods, is suggested by the formula written by 4 students, $C = \frac{5}{9}F - 32$, which was probably obtained by trying first to balance the equation by multiplying by 5 and dividing by 9 (in this way they eliminated the fractional number on the right side of the equation) and then they transposed the number 32. A mixed approach like this is often observed in pupils' approaches to equation solving, in particular when fractional numbers are involved. The same approaches were observed during the post-evaluation. It is worth considering that between the pre and the post-evaluation there was no teaching on rearranging formulae.

In the pre-evaluation, 7 out of 12 of the UK students produced correct responses and 5 of these gave their solution in the following form $C = \frac{F - 32}{9/5}$ which suggests that they have first subtracted 32 from both sides of the equation and then divided both sides of the equation by the whole fraction 9/5. When asked how they rearranged equations they talked in terms of "doing the same thing" to both sides of the equation. These students would first have met rearranging equations in mathematics using a "balance" metaphor. They were happy to leave their answers in this form (i.e. with 9/5 in the denominator) because very little emphasis is currently placed on simplifying algebraic expressions in the UK. Only 7 of the same 12 students (not the same set of 7 who obtained correct answers to this item) continued to study mathematics at A-level and these students were introduced to a new approach to rearranging formulae in the context of finding inverse function, using a method for unchaining a composite function. Of these A-level maths students, 5 who obtained correct solutions in the pre-evaluation produced incorrect responses in the post-evaluation. The incorrect responses were in the following form $\frac{5}{9}F - 32$. We conjecture that these new incorrect responses were influenced by the new approach which they had recently been taught in mathematics. This conjecture is strengthened by the fact that the two students who had produced correct responses in the pre-evaluation, but who no longer studied mathematics, still produced correct responses in the post-evaluation.

Reading a Numeric Pattern. The following item was used to probe students' approach to reading numeric patterns from tables.

The following table shows the increase of population (in millions).

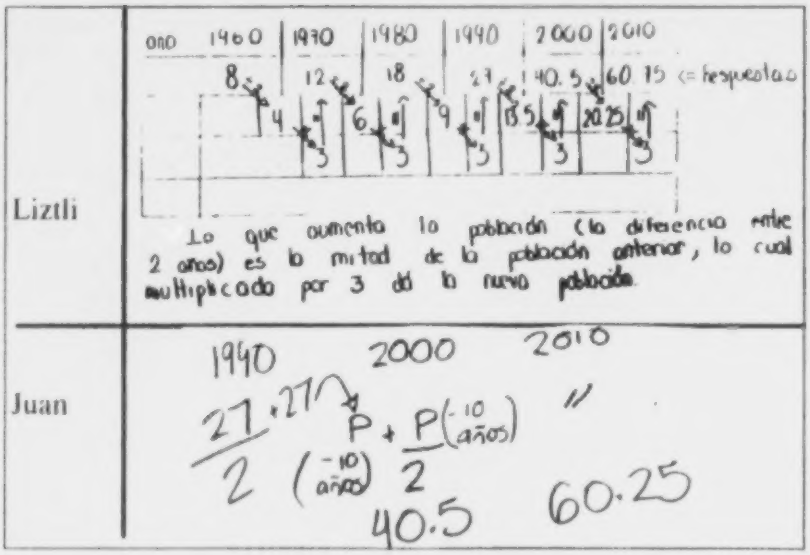
Year	*1960*	*1970*	*1980*	*1990*
Population	*8*	*12*	*18*	*27*

Can you predict the population for the year 2000 and 2010?

It has to be stressed that in contrast to the UK students who had previous school experience in pattern recognition, this was not part of the Mexican students' background.

Four different strategies for solving this problem were observed in the Mexican students' responses to this item during the pre-evaluation: the constant ratio method (2/9, both correct); the 1st difference method (2/9, one incorrect, one incomplete); the 2nd difference method (1/9, correct); approximate solution without explicit calculations and guided by the context of the problem (3/9). In contrast with the Mexican students the majority (11/12) of the UK students in the pre-evaluation attempted to use a difference method and only one student used a constant ratio method. This student and 6 of the others obtained appropriate answers. The UK students were explicitly taught difference methods in their pre-16 mathematics, which was not the case for the Mexican students. It could be argued that the constant ratio method is more appropriate to the scientific context of the problem, and interestingly the only UK student who used this approach was also the only student to include units in his answers, which suggests a focus on the physical situation.

In the post-evaluation, the majority of Mexican students (6/9) used the constant ratio method. They used different approaches for deducing the multiplication factor. One of these is illustrated by the solution of Liztli (Figure 4). She used a tabular notation and finds that given the population of a decade she can obtain the one corresponding to the next decade by dividing the first by two and multiplying the result by 3.


Liztli
año 1960 1970 1980 1990 2000 2010
8 12 18 27 40.5 60.75 ⇐ respuestas
4 6 9 13.5 20.25
3 3 3 3 3
Juan
1990 2000 2010
27 + 27/2 (-10 años)
P + P(-10 años)/2
40.5 60.25

Lo que aumenta la población (la diferencia entre 2 años) es la mitad de la población anterior, lo cual multiplicado por 3 da la nueva población.

The increase of the population (the difference between 2 years) is the half of the previous population, which multiplied by 3 equal the new population.

Figure 4

Another approach is shown in Figure 4, Juan. Here the student realises that to calculate the new population he had to add to the previous one its half and he expresses this general rule by writing the expression $P + \frac{P}{2}$. Both of these approaches might have been influenced by their experience in working with spreadsheets. In fact, in the first one we observe a tendency to use a spreadsheet representation and to look for a rule that would allow the generation of the numbers of subsequent columns from the previous one. This might be seen as a recursive method. The second approach also shows a tendency to use recursivity, in fact, the new numbers are deduced by using the previously deduced ones. This approach could have been influenced by the spreadsheet work which had emphasised recursive solutions within the context of modelling problems. In the post-evaluation the majority of the UK students (8/12) used a constant ratio method (all correctly) and only 3 students used a difference method. Again we suggest that the constant ratio method is likely to have been influenced by the spreadsheet work.

Conclusions

Our studies with groups of Mexican and British pupils have shown that whereas there may be some invariant obstacles to algebraic thinking which go across cultures, there are also many ways in which the school mathematics culture determines the ways in which students approach algebra and mathematical modelling problems. Clearly this is a trivial statement in the extreme cases in which for example students have or have not been taught any algebra. These results put into question research on students' conceptions/misconceptions in algebra which only explain student differences in term of psychological development. On the other hand what our research in showing is that the whole notion of 'school algebra' has to be unpacked with more attention being paid to what is meant by school algebra in a particular school culture. This analysis has to look much further that the superficial names given in a curriculum document (for example "solving equations") on which many comparative work is based. It also has to take into account method of assessment and also teaching approaches.

References

Clement, J. (1982), Algebra Word Problem Solutions: Thought Processes Underlying a Common Misconception. Journal for Research in Mathematics Education, 13, pp. 16-30.

Hart, K.M., Brown, M.L., Küchemann D.E., Kerslake, D., Ruddock, G. and McCartney, M. (1981), *Children's Understanding of Mathematics: 11-16.* John Murray Publishers, London.

Küchemann, D.E. (1980), *The Understanding of Generalised Arithmetic (Algebra) by Secondary School Children.* PhD Thesis, University of London.

Lave, J. (1988), *Cognition in Practice: Mind, Mathematics and Culture in Everyday Life.* Cambridge University Press, Cambridge, England.

Sutherland, R., Rojano, T., Mochón, S., Jinich, E. and Molyneux, S. (1996), *Mathematical Modelling in the Sciences Through the Eyes of Marina and Adam.* Paper submitted to PME-20, University of Valencia, Valencia, Spain.

Vygotsky, L.S. (1978), Mind in Society: The Development of Higher Psychological Processes, Harvard University Press, Cambridge, Massachusetts.

We want to acknowledge the Spencer Foundation for the support given to the project: The Role of Spreadsheets within School-based Mathematical Practices (Grant B-1493). Also thanks to staff and students of Escuela de la Ciudad de Mexico and Hampstead School, UK.

ON THE INTRODUCTION OF REAL NUMBERS IN SECONDARY SCHOOL. AN ACTION-RESEARCH EXPERIENCE

Romero, I. (Granada University. Spain). Rico, L. (Granada University. Spain)

ABSTRACT: This paper is aimed to present some of the main results obtained in an action-reserach study carried out with a class of children aged 14-15, in order to detect difficulties and potentialities for the understanding of real numbers at an initial stage. Our approach is of curricular type, and is based in a didactical process which rests in the simultaneus and complementary use of the different representation systems, symbolical and graphical, of the real numbers; it also considers key problems in the historical development of the concept. Interactional aspects are relevant to this work, which focuses in conlict and discussion.

I. INTRODUCTION

The present work has been done within a research line in Mathematics Education named *Numerical Thinking*, which studies the teaching, learning and comunication fenomena of numerical concepts in the Educational System and in society. The Numerical Thinking deals with the different cognitive and cultural processes by which human beings asign and share meanings using different numerical structures (Castro, 1994). It´s an inquiry line which tackles this study from a triple perspective. On one side, it deals with specific numerical structures; secondly, it studies the cognitive functions that human beings develop through the use of numerical concepts and properties; in third place, the Numerical Thinking looks after the problems and situations which can be approached and solved by means of the considered numerical structure and, specially, it analyses the phenomenology that underlies the above mentioned structure.

From this perspective, our work can be framed in the conceptual field of *Real Numbers*, in the sense stablished by González(1995): set of concepts, procedures, and relationships which constitute the mathematical structure of real numbers, toghether with the activities and cognitive functions that characterize the ways of use of the concepts, procedures and relationships of the above mentioned numerical system, and with the group of phenomena and situations that admit to be analysed through them and the problems that can be approached with them.

In general terms, we have noted an scarcity of references in the research literature above the subject. We have found reports over students´intuitions on the concept of real number, but there is a lack of researchs with curricular orientation which study in depth the development of such intuitions. Among the works directly related to the topic, we can note different orientations:

-Works over the didactical epistemology of real numbers (Tall and Schwarzemberger, 1978; Arsac, 1987; Arcavi and Bruckheimer 1987; Burn, 1990).

-Analysis of students´conceptions over real numbers (Robinet, 1986; Monaghan, 1988; Romero i Chiesa, 1994; Fischbein, 1995).

-Studies with didactical processes about real numbers (Douady, 1980; Corial et al., 1993)

-Woks over students´conceptions about infinity (Fischbein and Tirosh, 1979; Gardiner, 1985; Moreno and Waldegg, 1991; Tall, 1980).

At this point, authors like Monaghan (1988) and Fiscbein (1995) demand further research in this area, because proyects carried out till the moment haven´t analysed the utility of a conflict-discussion approach departing from students´initial conceptions. Our work is aimed to explore this proposal. More precisely, *our aim was to explore potentialities and difficulties ocurred when introducing the concept of real numbers to pupils 14-15 years old by means of a didactical proposal characterized by:*

-Assuming the complexity of the concept and opening ways for the presentation, understanding and solution of key problems in the construction of the above mentioned concept.

-Being based in a complementary a simultaneus work with the symbolic and grafic systems of representation of the real numbers.

-Being framed in a curricular context, considering both, the limitations and the possibilities which the classroom , as a natural complex setting, provides.

2. METHODOLOGY

It is evident that this approach to the study of didactics of real numbers, departing from a curricular contex, is complex and presents a number of problems. The social complexity of the teaching-learning-process, rich and dynamic as it is, can be considered as a problem, as well as an aditional potentiality.

In our particular case, we added another important factor: the decision that one of the authors (Romero in this case) were the teacher of a class of 14-15 years-old pupils in which the experience was to be carried about, assuming the double role of teacher and researcher. There were various reasons which led us to this decision:

-The current curricular treatment of the topic was far from an authentic conflict/discussion approach, which was, in our opinion (and according to the experts) the only meaninful way of dealing with the topic.

-The aboved mentioned approach posed a number of problems to be assumed by another teacher.

-The direct work with the students was an advantage for the implementation and development of our ideas. If we bear in mind that it was an exploratory study and that we considered it as a dynamic and evolving process, the fact that the researcher was an expert in the content was key to adapt herself to the unpredictable and continuous evolution of the process in matter of content and understanding, in order to get the principal aim of detecting points of potentiality and difficulty in the construction of the concept of real number.

In this way, as we have already mentioned, the presenting author had a role of teacher researcher. The notion of teacher researcher has various possible meanings. Richardson (1994) distiguishes between the notion of teacher researcher in the context of what she calls *practical inquiry* and the notion of teacher researcher in *formal research*. The main difference between this two types of research is that practical inquiry is conducted in one´s everyday work life for purposes of improvement and there is an inmediate concern for finding practical solutions to problems which arise in everyday reality; formal research is designed to contribute to a general knowledge and its results must be presented, contrasted and valued within the scientific community. We place ourselves in the context of formal research, being our case that of a *researcher expert in content and didactic of the discipline as teacher*.

In this conditions, the study was designed as an action-research experience. The general structure is based in the stages that have been described as basic in a action research process: Planification, Implementation, Observation and Reflexion, as well as in the idea of sequentiality in cycles (Lewin, 1946; Kemmis, 1982; Elliot, 1973; Ebbut, 1985; Whitehead, 1984; quoted by McNiff, 1988; Castro, 1994). Our particular scheme was the following:

CYCLE 1
    1.1. Identification of the general research problem.
    1.2. Development of the research problem: theoretical analysis.
    1.3. General plan:
        .Phase 1: Rational Numbers.
        .Phase 2: Irrational Numbers and real Numbers.
    1.4. Implementation of phase 1.
    1.5. Observation and effects of phase 1.
    1.6. Reflexion upon phase 1.
CYCLE 2
    2.1.Revision of the general idea.
    2.2.Readjustment of the general plan.
    2.3. Readjustment of phase 2.
    2.4. Implementation of readjusted phase 2.
    2.5. Observation and effects of phase 2.
    2.6. Reflexion upon phase 2.
FINAL CONCLUSIONS OF THE STUDY

In order to define, articulate and specify our general research problem (stablished in 1.1), we made a detailed study of the two first component of the conceptual field of real numbers: the mathematical structure, and the cognitive competencies or functions associated to the field (point 1.2).

To deal with structural aspects we made an analysis of the historical evolution of the concept of real number, focusing our attention on the obstacles and conceptual breaks detected in the historico-critical reflexins about the field (Artigue, 1990; González, 1995); we also considered in detail the different systems of representation which have been constituted trhough the history of the concept and the problems implied en each system.

To deal with cognitive aspects, we based on the notion of understanding by Hiebert and Carpenter (1992); we focused on the systems of representation of real numbers, their elements and relationships, in order to explore the use that students make of each of these systems, the meaning they attribute to them and the contexts in which they employ them, as well as the contradictions and conflicts that arise when working with the different systems and the relationships among them.

With respect to the third component of our conceptual field, which refers to the phenomenology of the implied concepts, it has´n been explicitly incorporated to our study; that is, we haven´t made a systematic analysis of the phenomena, problems and situations charasteristic of the real numbers field. Nevertheless, we have made use of a number of such problems in our work with the pupils.

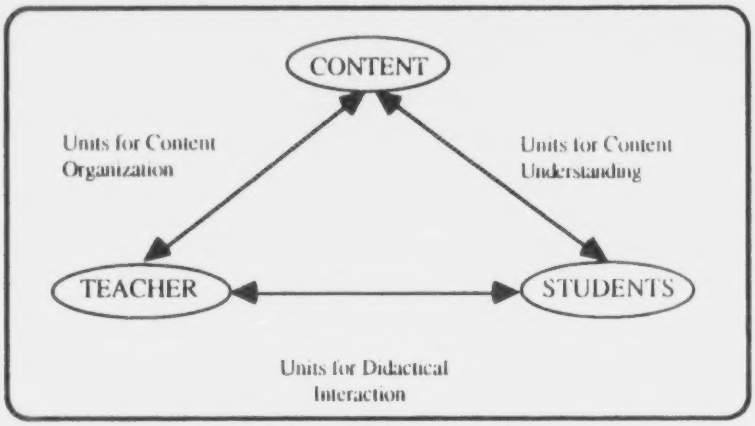
Besides, we have paid an special attention to social and contextual aspects in the process of knowledge construction.

Once we had made a detailed analysis of the research problem, we designed a didactical plan to be implemented with a class of children aged 14-15. For that, we stablished two research foci: the first focus was on the didactical problems and potentialities derived from the symbolic notations of real numbers, and the second focus was on the conflicts and potentialities arised in the work with geometrical representations and the relationships between the different representation systems. Both foci were broken down in a number of specific subobjectives (4 subobjectives for the first focus and 5 for the second), which were made operative by means of various Research Questions (8 for the fist focus and 15 for the second); these questions were proposed to the pupils as school tasks, exercises, working situations, exploration works, exams or questions for discussion, that constituted the axis around which the didactical process was articulated.

After this, we proceeded with the implementation of the designed plan with a class of thirty two 14-15 year-old children in a Secondary School, being the author of the paper I. Romero the teacher of the group.

Following our action-research scheme, we executed the consecutive stages of implementation, analysis of the effects and reflexion for each of the planned phases: phase 1 (corresponding to Rational Numbers) and phase 2 (corresponding to irrationals and Real Numbers).

The data were obtained from different sources: students´written documents, teacher´s diary, audio and video tapes of classroom discussions. They were interpreted, following a qualitative methodology, by means of Analysis Units, constructed by the authors for reporting about the relationships between the components of the didactical triangle Students-Content-Teacher:


CONTENT
Units for Content Organization
Units for Content Understanding
TEACHER
STUDENTS
Units for Didactical Interaction

From the analysis of these data, we have obtained information about different problems and potentialities that were present when working with the students (14-15 years old) on the concept of real numbers at an initial stage, and the way they evolved along the course of the didactical process. In the next section we present some of these results.

3. RESULTS

We´ll classify the information obtained attending to those aspects in which a meaningful progess in pupils´understanding could be observed and those ones that revealed as specially difficult points in the construction of the concept of real number.

Some of the most meaningful advances in pupils´understanding detected were the following:

-Typology of possible decimal expressions employed by the students, meaning they attributed to them and discrimination among different types of real numbers by means of the decimal typology. At the begining of the process, only 50% of the pupils recognise the existence of infinite non-periodical decimals and around 30% accept

that these expressions correspond to numbers (for some of the studens a square root is an operation); the examples given correspond to square roots, some infinite decimals with arbitrary digits and a few mentions of π; the origin of these decimmals is attributed to square roots and, in the case of π to a fraction (lenght of the circunference / diameter).

By the end of the didactical process around 80% of the students consider that infinite non-periodical decimals are numbers, around 30% specify different types: square roots, π, ϕ (the golden ratio), and infinite decimals with arbitrary digits (following a rule or not), and they can correctly classify the different types according to its origin which has became meaningful to them by working with the different types of numbers in classroom situation or in exploring works.

-The use of different representation of real numbers according to the desired aim, and the use of the correspondence between periodical decimals and fractions to justify, by analogy, certain properties of infinite non-periodical decimals corresponding to construible irrationals. It´s interesting to note how the pupils resorted to different representations of the real numbers according to different objectives, and how they were able to make explicit that you need to use use fractional representation in order to do arithmetic with periodical decimals, and you need to do the same to represent them "exactly" on the number line. After that , they have resort to the analogy periodical decimal/fraction - infinite nonperiodical decimal/radical to argue that you can do arithmetic with non-periodical decimal by using their radical expression, and the same for representing them "exactly" on the number line. (Although some children questioned the identification between different representations and the aboved mentioned analogy, as we´ll comment in the difficulties section).

-The notions of conmensurability and unconmensurability Conections with the correspondence between the real numbers and the number line For these students wasn´t easy to understand the conept of unconmensurability of certain lengths with respect to the unit of measure, departing from their decimal expression, infinite and non-periodical and, so, non correspondent to a fraction. But the work on classroom situations in the geometrical context alowed a numer of students to construct a common languaje which permited them to asign the term "proportion" to unconmensurable ratios and to used it to solve problematic situations as, for example, the understanding and classificaton of the Golden Ratio by comparing it with the ratio of the length of the circunference to its diameter.

When pupils were asked to discuss the argument that "as two lengths always have a common measure (and so they colud be expressed by means of fractions), all the points on the line were rationals (because you consider the length from the point to the origin and measure this length with the unit lenght on the line)", 60% of the pupils desagreed giving coherent arguments, and 26% refuted the given argument in terms of the existence of "proportions", which corresponded to irrational numbers.

Some of the most meaningful difficulties in pupils´understanding detected were the following:

-*Certain aspects of the correspondence between operatory notation and decimal notation of real numbers, and the construction of the concept of real number supporting on its different symbolic representations and the relationships between them.* Various difficulties in the conversion fraction-periodical decimal and viceversa were diagnosed. These difficulties make that this corespondence was stablished in an intuitive way in most of the pupils. The lack of consistency and systematic reasoning to support it is notorious at this stage, and the same occurs in the case of the correspondence between the decimal and operatory notation of the irrationals, which is also stablihed by many of the pupils (68%), but an intuitive and non-consistent way; a number of problems were diagnosed here too.

-*The conflict between the actual finitude of centain lengths uncommensurable with the unit of measure and the potential infinitude of their decimal expression.* The conflict already arised in the case of periodical decimals. There were pupils who manifested their difficulty in accepting that a periodical decimal could be represented "exactly" on the real line by means of a fraction, because having infinite digits it could never be represented "exactly", and then they couldn´t admit that an infinite decimal and a fraction were "the same thing".

Also, the students find very difficult to accept that the side of an square of area 3 has a finite length and that its decimal expression is infinite and non-periodical; for that reason the side of the square cannot measure $\sqrt{3}$ "exactly", for them. Even a girl refused to connect on side of the square to the other because she wanted to paint $\sqrt{3}=1{,}7320508$...and, as it was infinite, she couldn´t finish the length, she couldn´t reach the final point of conection.

We find that this conflict is perflectly reasonable, although it´s solution is out of the reach of pupils at this stage, because it implies the step from considering an infinite decimal expression as a process to consider it as an actual thing, in terms of actual infinity. Nevertheless, we consider important enough the arising of the conflict in the process of constructing an understanding of real numbers.

REFERENCES

Arcavi, A. , Bruckheimer, M. y Ben-Zvi, R. (1987). History of Mathematics for teachers: the case of irrational numbers. *For the Learning of Mathematics*, 7, 18-23.

Arsac, G. (1987). L´origine de la démonstration: Essai d´epistémologie didactique. *Recherches en Didactique des Mathématiques*, 8, 267-312.

Artigue,M. (1990). Epistémologie et Didactique. *Recherches en Didactique des Mathématiques*, 10, 241-286.

Castro Martínez, E. (1994), *Exploración de Patrones Numéricos mediante Configuraciones Puntuales. Estudio con escolares de Primer Ciclo de secundaria (12-14 años).* Granada: Tesis Doctoral.

Coriat Benarroch, M. y otros (1993). *Numbers and colours. International Journal of Mathematics Education, Science and Tecnology*, 24, 501-510.

Douady, R. (1980). Approche des nombres réels en situation d´apprentissage scolaire (enfants de 6 à 11 ans). *Recherches en Didactique des Mathématiques*, 11, 77-110.

Euclides (1991). *Elementos. Libros i-IV.* Madrid: Biblioteca Clasica Gredos.

Fischbein, E. , Tirosh, D. y Hess, P. (1979). The intuition of infinity. *Educational studies in Mathematics*, 10, 3-40.

Fischbein E., Jehiam, R., Cohen, D. (1995). The concept of real number in High-School. Students and prospective teachers. *Educational Studies in Mathematics*. 29, 29-44

Gardiner, A. (1982). *Infinite Processes*. New York: Springer-Verlag.

González Mari, J.L. (1995). *El campo conceptual de los números naturales relativos.* Granada: Tesis Doctoral.

Hiebert, J. y Carpenter, T. (1992). *Learning and teaching with understanding.* En *Handbook of Research on Mathematics Teaching and Learning.* Grouws, D.A. (Ed.). New York: MacMillan Publishing Company.

Janvier, C. (Ed.) (1987). *Problems of representations in the teaching and learning of mathematics.* London: Lawrence Earlbaum Associated, publishers.

McNiff, J. (1988). *Action Research: Priciples and practice.* London: Macmillan Education Ltd.

Monaghan, J. (1988). Real mathematics. *The Mathematical Gazette*, 72, 276-281.

Moreno, L. y Waldeg, G. (1991). The conceptual evolution of actual mathematical infinity. *Educational Studies in Mathematics*, 22, 211-231.

Rico, L. (1990). Diseño Curricular en Educación Matemática. Elementos y Evaluación. En *Teoría y Práctica de la Educación Matemática*. Llinares, S. y Sánchez, V. (Eds.). Sevilla: Alfar.

Robinet, J. (1986). Les réels: quels modéles en ont les élèves? *Educational Studies in Mathematics*, 17, 359-386.

Romero Albaladejo, I. (1993).*"La introducción del Número Real en Educación secundaria"*. Memoria de Tercer Ciclo. Programa de Doctorado de la Universidad de Granada.

Romero i Chesa, C. Una investigación sobre los esquemas conceptuales del continuo. Ensayo de un cuestionario. Pendiente de publicación en *Enseñanza de las Ciencias*

Tall, D. y Schwarzenberger, R. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, 82, 44-49.

COUNTING, ESTIMATION AND THE LANGUAGE OF UNCERTAINTY

Tim Rowland

Homerton College, Cambridge

The empirical study summarised in this paper arises from earlier study of 'hedges' in mathematics talk. Hedges are words and phrases which make the truth conditions of an utterance less precise. In the mathematics classroom, hedges are an important tool in the student's linguistic repertoire e.g. for protection against accusation of being wrong. This study, based on interviews with 230 children aged 4 to 11, was designed to trace the development of children's competence and tendency to hedge.

BACKGROUND AND PURPOSE

Ishka, an 11-year-old girl, was invited to assent to a combinatorial prediction by another pupil. Ishka was more than somewhat equivocal in her reply: "Well maybe not exactly, but it's around fifty basically?". In mathematics talk, students make frequent recourse to such 'hedges', for a number of reasons, and often as an indicator of uncertainty (Rowland, 1995).

Hedges include words such as *sort of, about, approximately* - words which have the effect of blurring category boundaries or otherwise-precise measures - as well as words and phrases such as *I think, maybe, perhaps*, which hedge the commitment of the speaker to that which s/he asserts. Epistemic modality (mainly achieved in English by the use of modal auxiliary verbs such as *may, might*) is another important means by which speakers mark tentativeness about their assertions (Coates, 1983).

The study summarised in this paper focuses on pupils' recourse to hedges and modals when presented with tasks which invite counting or estimation, Wiener (1972) identified a vicious circle in children's attitudes to estimation: poor estimators, not surprisingly, viewed estimation as "risky", avoided it, and remained poor at estimation. Clayton (1992) develops this affective theme, presenting estimation as a risk-taking activity.

The language of approximation may be deployed in order to achieve one or more of a number of pragmatic goals (Channell, 1994, pp. 173ff) and interpreted by reference to the observance of a cooperative principle and maxims of conversation due to Paul Grice (1989). This is elaborated in Rowland (*op. cit.*).

In their first two or three years at school, young children are conditioned to receive a "How many?" question as an invitation to *count* rather than to estimate. Counting a small set may be regarded as a less risky enterprise than estimating its cardinality. If this is the case the young child's response to such a question is less likely to be modalised or hedged than the older child's. Furthermore, the young child will not be able to hedge until s/he has learned how to achieve that effect with language.

This study was designed to test the hypothesis that modal forms and hedges will be relatively absent in mathematics talk in early childhood, and that thereafter one can discern progressive development of modal/hedging capability and use in individuals through the years of primary (elementary) schooling.

METHOD

An empirical study was carried out in a 4-11 primary school. There were 230 children on the school roll. Every child was asked the same three "How many?" questions (see below) in private, one-to-one interview. The object was to test the expectation that the language of modality and hedging will be more commonplace among the oldest children (10-11) than the youngest (4-5), with some sort of continuum evident between these extremes.

The fieldwork centred on the following three tasks:

> Task 1: The interviewer produces a plate on which 19 coloured sweets have been placed so that each is visible. The child is asked, "Can you tell me how many sweets there are on the plate?".
>
> Task 2: The interviewer produces a high-quality colour photograph of a small glass containing 14 sweets. The child is asked "Can you tell me how many sweets there are in the glass?"
>
> Task 3: The interviewer shows the child two thin plastic tubes (both are about 25 mm in diameter and 10 cm high). One contains 10 sweets, the other 20. The interviewer says "There are ten sweets in this tube (indicates). I know that, because I counted them when I put them in. Can you tell me how many sweets there are in this (indicates the other) tube?"

In the first task, the child can actually count the sweets if s/he chooses to do so, but may also make a reasonable estimate if s/he so chooses. The second task was designed so that the precise number of sweets in the glass was indeterminate. It can not reliably be determined by counting, since not all of the sweets are visible in the photograph. Some kind of estimate is therefore necessary. Similarly, in the third task, the precise number of sweets cannot be determined by counting, since not all are visible. However, the height of the sweets in the second tube, relative to the first, may provoke an elementary form of proportional reasoning, namely doubling. The interviewer sought to understand whether any such strategy and inference was a factor by means of probes such as "How did you know that?".

RESPONSES AND CONTINGENT QUESTIONS

For all three tasks, each child was asked to say how many sweets there were (respectively on the plate, in the picture, in the second tube). Two kinds of response were categorised as 'Marked':

- those responses which conveyed vagueness through specific linguistic hedges - "I think there are ten", "About ten", and so on;
- statements of possibilities or conjectures, with modal auxiliaries e.g. "it might be ten".

The label 'Marked' and derivative forms will consistently be highlighted in this paper with a capital letter as a reminder of its current, if interim, technical meaning referring to the two itemised response-types. Hedges and modals will be described jointly as Markers.

If one of these two kinds of Marker was spontaneously present in the initial response of the child, the interviewer noted it and moved on to the next task (or concluded the interview). Such a spontaneous hedge or modal was denoted a *primary* Marker. If, on the other hand, the primary response was un-Marked (e.g. "There are nineteen" or simply "nineteen"), the interviewer would ask a supplementary question, "Do you think there are exactly nineteen (or *n*)?". If this second question provoked a Marker in the child's reply, then this *secondary* Marker was recorded. Thus, for each of the three tasks, primary and secondary Markers were mutually exclusive.

DATA

Each interview either audio-taped or videotaped. The children's responses were recorded on a proforma and entered onto a database. The software enabled the usual data-interrogation methods. Some data relevant to this paper are displayed below. The results on Marking are presented here in four age-bands, related to the first seven 'Years' (grades) of schooling in England and Wales:

Year R (reception):	the first full year in the school, the oldest child being at most 5 years 9 months at the time of the interviews.
Years 1 and 2:	'Infants', aged between 5 years 9 months and 7 years 9 months.
Years 3 and 4:	'Lower Juniors', aged between 7 years 9 months and 9 years 9 months.
Years 5 and 6:	'Upper Juniors', aged between 9 years 9 months and 11 years 9 months.

The numbers of children in these four bands was 45, 70, 65 and 50 respectively. The numbers of children giving a Marked response to each question are presented below as percentages of the number in each band, so that comparisons between the bands may be made.

		Task 1: Markers (% of Band)			Task 2: Markers (% of Band)			Task 3: Markers (% of Band)		
Band	N=	Primary	Secondary	All	Primary	Secondary	All	Primary	Secondary	All
R	45	9%	2%	11%	9%	2%	11%	2%	7%	9%
1 and 2	70	1%	6%	7%	1%	6%	7%	6%	6%	11%
3 and 4	65	2%	11%	12%	9%	18%	28%	26%	15%	42%
5 and 6	50	6%	20%	26%	28%	20%	48%	22%	36%	58%

Note that for each of the Tasks 1 to 3, primary (spontaneous) and secondary (provoked) Markers are mutually exclusive.

OBSERVATIONS

1 The cumulative Marked responses on Tasks 1 and 2 show a drop from the first band (Y R) to the second (Y1-2) with consistent increases thereafter. This cumulative decrease is the result of sharp decreases in primary Marking between those bands.

2 With regard to secondary Marking only (Tasks 1 and 2), there is a consistent rise from band to band over the whole age range.

3 On Task 3, the trend (minor inconsistencies apart) is of steady increase with age.

4 On the whole, there is a greater tendency towards secondary Markers rather than primary in the last three bands (Y1 to Y6), but for the reverse in the youngest (Y R).

INTERPRETIVE FRAMEWORK

It is reasonable to suggest that, in a broad sense, the data obtained from the tasks and interviews support the expectation that ability to use linguistic Markers, and the tendency to do so, develops with age - at least over the years of primary schooling.

I propose a socio-linguistic developmental interpretative framework which would account for the upward trend, and which might also accommodate the unanticipated initial drops in primary Marking (from band R to band 1-2) noted above in Observation 1.

I suggest that the modal and hedging linguistic behaviours of the children in each band are related to three fundamental developmental dimensions.

- The child's developing 'apprehension' of school - of the roles of the players (particularly teachers and their pupils) in the school situation, and the way that they relate to each other in learning situations. Being aware of the role-differences between home and school is part of what Berger and Luckmann (1967) call 'secondary socialisation'.
- The child's developing (confidence in his or her) ability to produce desired behaviours within a variety of school practices. In particular, the child progresses over the first few years of schooling from a position where counting is a significant challenge to one where it is a routine if necessary chore.
- The child's developing awareness of modal concepts and command of modal language. Hypothetical reasoning is, in the classical Piagetian formulation of cognitive development, a distinguishing hallmark of formal operational thinking. The first epistemically modalised statements in pre-school children tend to occur about six months later than deontic meanings, related to obligation and permission (Stephany, p. 396), but are still extremely rare in comparison.

INTERPRETATION OF THE DATA

Against this background I propose the following developmental narrative, to account for the trends in the data by reference to the interpretive framework above.

The account of changes in Marked language and performance in the primary years (age four to eleven) is presented below in terms of three developmental phases, which I have termed Initiation; Suspicion; Approximation and Protection. Each of these phases will be briefly characterised, and illustrated by extracts from the 230 task-based interviews.

YEAR R - INITIATION

The child's apprehension of school is relatively naive, and adult behaviour - questioning in particular - is taken at face value, without suspicion. Counting is a relative novelty, and the child is aware that her performance is sometimes faulty; this is hardly surprising, given the complexity of the process, as analysed by Gelman and Gallistel (1978), Fuson (1988) and others. Whilst the task of enumerating 19 items may be *accessible* to a Year R child, a teacher will alert the child even when the count is substantially competent yet not entirely accurate; because accurate counting is a major goal, a targeted skill, in this phase of schooling. Moreover, Gelman (1977) suggests that only about one five-year-old in six can *accurately* enumerate a set of 19 items, given one minute to do so. The child (Year R) may well wish, therefore, to acknowledge to the interviewer, (with a primary Marker) some doubt about the answer s/he gives. The child's Marking can be understood as straightforward cooperation, in effect observing Grice's maxim of Quality.

In each of the following transcript extracts, 'I' is the interviewer, 'C' the child.

Example 1: Boy aged 5:9 [M39]

M39:1 I: Can you tell me how many sweets there are on the plate?

2 C: [*counts aloud, points to each sweet in turn*] One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, .. twenty-three, .. twenty-four.

3 I: Twenty-four? Do you think there are exactly twenty-four?

4 C: Maybe not.

5 I: Why do you say that?

6 C: Because I might have counted two double.

This young boy uses Marked language fluently. He receives the "How many?" question as a straightforward invitation to count. He has command of the stable-order principle (Gelman and Gallistel, 1978, pp. 77-82) [M39:2], but applying the one-one principle to a disordered set of this size presents difficulties for him. He does not touch the sweets one by one, but points to them - a partial internalisation (Fuson, 1988, pp. 85-6). His ability to partition the set (*ibid.*) into counted and to-be-counted subsets is faulty *and he knows* it is sometimes faulty [6]. His attitude is something like: "That's what I make it but I know from experience that I may have made an error. That's simply the way things are when, like me, you're a novice at counting".

YEARS 1 AND 2 - SUSPICION

The child's apprehension of school includes the sense of being scrutinised by curious adults, of the existence of 'testing' questions. S/he is now expected to enumerate small sets routinely. The result is a manifest reluctance to use primary Markers in response to the first two tasks (perhaps because "he wants to know if I can get it right"), but the interviewer's probe ("Exactly?") may release an acknowledgement of uncertainty, using the same Plausibility Shields (predominantly 'I think' with a few 'maybe's) as the Year R child.

Example 2: Boy aged 7:5 [reference number 55]

The transcript is chosen for absence of Marked language.

M55:1 I: Can you tell me how many sweets there are on the plate?

2 C: [*quickly, touching each sweet*] One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, .. oh, [*restarts counting, now slower, placing sweets on the table whilst counting*] One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen.

3 I: And do you think there are exactly nineteen?

4 C: What?

5 I: Do you think there are exactly nineteen?

6 C: Er .. [*pause, then recounts into hand*] One, two, three, four, five ... One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen.

7 I: So do you think there are exactly nineteen?

8 C: Yes.

His first recount [M55:2] is presumably a self-correction after he suspects that his partitioning has gone wrong. There is no problem with the order of the tags, and the count is in fact accurate. The interviewer's probe [3, 5] is immediately taken to be a suggestion that he has mis-counted. Instead of hedging, he re-counts the set [6].

YEARS 3 TO 6 - APPROXIMATION AND PROTECTION

There is developmental continuity within this phase rather than qualitative change. The account which will follow characterises a child who has moved some way along that developmental continuum.

The child has a well-developed apprehension of her role in the practice of education. S/he realises that teachers' - including researchers' - questions about mathematics are usually not simple requests for information. On the other hand, the child is confident in her ability to count. If such a child does in fact count the sweets in Task 1, then a primary Marker is unlikely. On the other hand, s/he may judge that the interviewer is not interested in the precise number of sweets on the plate, and offer a primary Marked estimate (there is a corresponding rise in the fourth band), or a response which is tagged (by intonation) with a question mark.

S/he recognises that some quantities are indeterminate; on Tasks 2 and 3 s/he will realise, despite the awareness of 'testing', that the interviewer cannot expect her to give precise answers to these "How many?" questions. If her first answer is un-Marked, s/he will readily admit to uncertainty when asked if that answer is exact. S/he may recognise the proportional reasoning Task 3 for what it is, and (correctly) have some confidence that there are exactly 20 sweets in the second tube. S/he has developed competence to deploy Approximators such as 'about' (as well as modal auxiliaries) as epistemic markers, to introduce vagueness for protective purposes.

Example 3: Girl aged 10:6 [reference number 222]

M222 I: Can you tell me how many sweets there are on the plate?

9 C: [*doesn't count, takes 2 seconds to reply*] **About** twenty?

10 I: Now, can you tell me how many sweets there are in the glass?

11 C: [*doesn't count, takes 1 second to reply*] Ten.

12 I: And do you think there are exactly ten?

13 C: No! [laughs] **not exactly**.

9 I: OK. Now, I've put ten sweets in this tube ... can you tell me how many sweets there are in this tube?

10 C: [*stares, compares tubes, takes 1 second to reply*] Twenty.

11 I: And do you think there are exactly twenty?

12 C: **About** twenty-five ... **or** twenty.

13 I: And what makes you say that?

14 C: 'Coz it **looks like** half, twice as much as in there.

This pupil knows exactly when an estimate will suffice and, indeed, when nothing else is possible. For Task 1, she instantly judges that an estimate will meet the requirement of the Grice's maxim of Quantity. Her approximator "about" qualifies a suitably round number [2] with the force of a root hedge (as if to say "This is as much as you need to know"). She just laughs at the suggestion that her choice of ten [4] should be taken to be anything other than an approximation. For Task 3, she is explicit that proportional reasoning is the basis of her rounded estimate.

MODAL AUXILIARIES

Initially, hedges were to have been the sole focus of study, but the epistemic similarities between modals and hedges motivated the inclusion of modal auxiliaries. The two classes of language clearly overlap in adverbial forms such as 'possibly', 'maybe'. Only 19 children in the sample of 230 used modal language that was not also recorded as a hedge. These 19 were predominantly in the third age-band (Year 3 and 4) and in nearly every case the modal verb used was 'might', as in "There might be some more on the other side" (Year 3 boy, Task 2). See also [M39:6]. It is interesting to note that, whereas occurrences of *may* and *might* were rare, in every case they arose as secondary Markers, in response to the interviewer's prompt.

SUMMARY

I began this study with the expectation that the data would support a hypothesis that the ability to use linguistic Markers, and the tendency to do so, increases consistently through the primary years. These Markers (hedges and modal auxiliaries) serve epistemic and root purposes (Coates, 1983, p. 18) conveying either the speaker's uncertainty or their awareness that an estimate was appropriate (in fact, essential in the case of the last two tasks). The interpretive framework offered here is an account of some developmental aspects of conveying uncertainty in mathematics, including social aspects which would account for a dip in unprompted Marked language shortly after the child's initiation to schooling. Whilst it is clearly the case that the use of Marked language has, in some sense, to be learned, it is not so clear how that particular linguistic competence to convey propositional attitude is acquired. The unexpected outcome of this particular study, in the context of estimation activity, is that children may be socialised into suppressing this aspect of their linguistic competence until they discover, or assert, that - in some mathematics classrooms at least - it is alright to be wrong.

REFERENCES

Berger, P. and Luckmann, T. (1967) *The Social Construction of Reality*, Harmondsworth, Penguin.

Clayton, J. G. (1992) *Estimation in Schools*, Unpublished M.Phil. thesis, London, University of London.

Channell, J. M. (1994) *Vague Language*, Oxford, Oxford University Press.

Coates, J. (1983) *The Semantics of the Modal Auxiliaries*, London, Croom Helm.

Fuson K. C. (1988) *Chidren's Counting and Concepts of Number*, New York, Springer Verlag.

Gelman, R. (1977) 'How young children reason about small numbers', in Castellan, N.J., Pisoni, D.B. and Potts, G.R. (eds), *Cognitive Theory Vol. 2*, Hillsdale, New Jersey, Lawrence Erlbaum, pp. 219-238.

Gelman, R., and Gallistel, C. R. (1978) *The Child's Understanding of Number*, Cambridge, Mass., Harvard University Press.

Grice, H. P. (1989) *Studies in the Way of Words*, Cambridge, Mass., Harvard University Press.

Rowland, T. (1995) 'Hedges in mathematics talk: linguistic pointers to uncertainty', *Educational Studies in Mathematics*, 29(4), pp. 327-53.

Stephany, U. (1986) 'Modality', in Fletcher, P. and Garman, M. (eds), *Language Acquisition*, Cambridge, Cambridge University Press.

Wiener, B. (1972) 'Attribution theory, achievement motivation, and the educational process', *Review of Educational Research*, 42, pp. 203-215.

THE TRANSFORMATION OF MATHEMATICAL OBJECTS IN THE DIDACTIC SYSTEM: THE CASE OF THE NOTION OF FUNCTION

Luisa Ruiz Higueras, Departamento de Didáctica de las Ciencias
José Luis Rodríguez Fernández, Departamento de Didáctica de las Ciencias
University of Jaén, SPAIN.

ABSTRACT:

This paper forms part of some research work (Ruiz-Higueras, 1994a) in which a theoretical model for the notion of conception was developed, based on Vergnaud's idea of concept (Vergnaud,1991), which we applied to the characterization of conceptions about the notion of function both on a psychological and an epistemological level. In the experimental study presented here, we have attempted to demonstrate the gap which, all too often, exists between formally introduced concepts, both by school text books and by teachers in the classroom, and the knowledge effectively constructed by the pupils, as well as some of the didactic factors and phenomena which condition this fact.

1.INTRODUCTION

This paper forms part of some research work (Ruiz-Higueras, 1994a) which studies secondary school pupils' conceptions about the notion of function, using an epistemological and didactic analysis thereof. In the aforesaid research work, a theoretical model for the notion of conception was developed based on Vergnaud's idea of concept (Vergnaud,1991).

An experimental study was carried out aimed at analyzing secondary school pupils' conceptions on the function object using a sample group of secondary school pupils, completed by a study of the institutional relationships that are maintained with this mathematical object, taking as indicators official curricula, school text books and the notes taken by the pupils during the classes.

In this paper, we present an analysis of the answers given by the pupils to one part of a questionnaire which demonstrates the gap which, all too often, exists between formally introduced concepts, both by school text books and by teachers in the classroom, and the knowledge effectively constructed by the pupils, as well as some of the didactic factors and phenomena which condition this fact.

2.METHOD

2.1. Characteristics of the sample: A sample was chosen including 323 pupils belonging to four different secondary schools in the province of Jaén (Spain), three of which are located in the city and the other is in the country. The teachers from these schools did not have any previous experience of the questionnaire and the pupils had not been prepared to answer the questions.

2.2. Data collection techniques: We may classify the method used for collecting data as one involving the average (Dane, 1990). In our case, interest is centred on the personal relationship that the pupil has regarding the notion of function and the data analyzed are his/her answers - arguments, algebraic and graphical representations and procedures- to the items in the test. The technique for collecting data was a survey, the data were collected personally by the researcher, using a process of mixed interaction.

Table 1
BREAKDOWN OF THE PUPILS IN THE SAMPLE

COURSE	N° of pupils	Percentage
2nd year BUP (16 years old)	139	43.0
3rd year BUP (17 years old)	86	26.6
COU (18 years old)	98	30.3
Total	323	100

2.3. Process for constructing the questionnaire: The following phases were involved in the construction of the instrument:

1) Initial compilation of possible items to include in the test.

These items were taken from other research papers such as those by Vinner & Dreyfus (1989), Vinner (1983) and Tall & Bakar (1992), as well as exercises and problems included in text books from different study courses and didactic proposals drawn up by different bodies.

2) Selection of items to form the two pilot tests.

We took into account their greater or lesser suitability to the three components that we wanted to determine in the makeup of the subject's conceptions:

- invariants that the subject might attribute as essential notes that determine the function object;
- representations of this object;
- situations of variation in which the subject might (or might not) deem it appropriate to use the function object as a modelling tool.

3) Preparation and analysis of the items selected and their study by experts in this field.

4) Application of the test to a pilot sample and possible modifications.

5) Construction and application of the final test.

This involved a questionnaire made up by 6 items, including a total of 25 questions (Ruiz-Higueras, 1994a). During its preparation we opted for open questions instead of multiple choice type questions. We wanted to study the full range of possible lines of argument that the pupils were able to express when justifying their answers. This lines of argument would then allow us to deduce their conceptions about the notion of function. We should point out that we are mainly interested in highlighting the pupils' conceptions and not assessing their academic achievement.

2.4. Questions that we analyze in this paper: We presented the results of the analysis of the answers given by the secondary school pupils to one part of the previous questionnaire which includes the following questions:

If you had to explain to a pupil in the first year of the "BUP" course (15 years of age) what a mathematical function is, what would you say to him?
Would you show him some examples? Which ones?
Would you give him an exercise or a problem for him to solve? Describe one.

We focused these questions so that the pupils, in their answers, would attempt to give an explanation of the notion to their school pals in lower years. By this we intended the pupils to try to use all the resources that might be needed to explain sufficiently what they understand by mathematical function.

We were interested in studying the definitions that the pupils made up for the concept of function, analyzing, in detail, the mathematical elements that they included in their definitions. We observed the declaratory aspect covered by these definitions, since through them we were shown a very significant part of their personal relationship with the notion of function; the invariants that they associated with it in their decision, the representations that they used and the tasks for which they though it appropriate to use functions.

Furthermore, we analyzed whether they include terms such as:

-Application, correspondence, association;

-Transformation, dependence.

These sections have been included because we consider that they involve key terms for determining what is the notion of function. Application, correspondence, and association indicate in some way an assignation between objects, whereas transformation and dependence describe to us the effects of the variation (governed by laws or criteria) between changing objects: We may only perceive that one thing depends on another, with each one of them changing in order to verify what has been the effect of the variation and, therefore, of the transformation. Furthermore, we shall take into account the records (numerical, graphic or algebraic) that they include in their definitions.

We were also interested in knowing which tasks they proposed as exercises, as well as the functions that they selected for carrying out these exercises. This, together with the detailed study of the notes taken down by the pupils in the classes, allowed us to deduce some characteristics of the didactic contract (Brousseau, 1986) existing between the teacher and the pupils. We also compared the examples presented by the pupils with the definitions that they expressed for notion. The former would correspond to prototypes of the conceptual image (Vinner & Tall, 1981), whereas the latter would be an explicit statement of their definition of the concept. We tried to see whether or not there was any consistency between these two aspects, or whether the phenomenon of compartmentalization has occurred (Vinner & Dreyfus, 1989).

3. ANALYSIS OF THE RESULTS

Once the categories for analysis had been established, we proceeded to encode the data and record it in order to carry out a statistical analysis, using the SPSS and BMDP software packages.

Since the research is of a qualitative type, no marks were awarded to the different types of answers and neither did we add up the number of correct solutions for each pupil. On the contrary, we were interested in analyzing each one of the dependent variables separately as identified in the analysis of the questionnaire. For this reason, the statistical analysis was restricted to preparing comparative tables for each variable with respect to the academic year and the analysis of correspondences and cluster analysis for the joint study of the items that refer to the recognition of functions (Ruiz-Higueras, 1994b).

Table 2 shows the frequency and the percentage, with respect to the total number of pupils in each year and in the whole sample, with which different terms appear in the definitions proposed by the pupils for the notion of function. We should point out that the terms that appear in the Table are not mutually exclusive, since in almost all the definitions, the pupils include elements from more than one heading. For example, let's have a look at the following definition:

> *A mathematical function is an equation that we may represent graphically using curves and, by means of this representation we may study all of its characteristics.* (3rd year of BUP, 17 years old).

In this definition, the pupil used terms from the algebraic register -"an equation" and from the graphic register -"graphically represent"-.

After analyzing all the protocols, we observed that not one pupil managed to formulate the definition of function accurately. Even those who used the terms *application* or *correspondence* did not mention the existence of the domain or the image set, nor the need for uniqueness for the image elements. In general, the *personal definition* of these pupils differs from the formal definition that appears in their text books.

As Table 2 shows, the highest percentages correspond to definitions that include elements of the algebraic register (60.4%) and numerical register (37.2%). For the vast majority, function is an operation between numbers. Their definitions are really a "story" about the actions that they normally carry out for solving their exercises in class.

36.2% refer to graphical representation in their definitions. By analyzing their definitions we observed that they always include the graph as the end of an algorithm process. So, for example:

> *A function is a mathematical operation that consists of an equation in which the values are successively replaced in order to obtain a result on a table. We may represent this table graphically, thereby obtained a graph.* (COU- 18 years of age).

We may observe how in these kinds of definitions, the pupils did not consider the graph of a function to be the representation of the relationship that exists between the variables, nor did they analyze its characteristics. That is to say, they did not point out the display potential that the graph has for representing the overall properties of the function.

Table 2

FREQUENCY AND PERCENTAGE WITH WHICH DIFFERENT TERMS APPEAR IN THE DEFINITIONS PROPOSED BY PUPILS FOR THE NOTION OF FUNTION

	Course years			**Total**
Mathematical terms	2nd of BUP (16 years old)	3rd of BUP (17 years old)	COU (18 years old)	
Numerical	55 39.6	28 32.6	37 37.8	120 37.2
Algebraic	75 54.0	52 60.5	68 69.4	195 60.4
Graphics	43 30.9	35 40.7	39 39.8	117 36.2
Application	27 19.4	3 3.5	14 14.3	44 13.6
Transformation	7 5.0	0 0.0	6 6.1	13 4.0

The terms *application, correspondence* or *association* were used by only 13.6% of the pupils. Nevertheless, both in their textbooks and in their class notes, the definition of function that appears is fully labelled as an application between two subsets of R. In order to back up this definition, both the teachers and the school text books adopt an intuitive option based on Venn diagrams. This device allows them to locate the notion of function in continuity with previous notions already learnt by the pupils, such as correspondences. Nevertheless, this device often leads them towards an excessively trivial didactic treatment of this mathematical object. The pupils also used diagrams in an attempt to support their definition with an "ostensive" foundation which provides them with security. In these representations they establish correspondences exclusively between natural numbers. The saving offered by natural numbers in the numerical calculation makes it possible that they may in fact constitute an obstacle for finding out about real variable functions (whose domains are subsets of R); since most of the pupils considered them to be the only possible numerical domain, thereby restricting the notion of function exclusively to that of succession.

Just 4% of the pupils considered a function as a transformations o a change between variables. When a function is considered as a variable, the nature of the change is accepted. The concept of variable is precisely that capacity in the mind to characterize this change. It is obvious that, for our sample of pupils, the presence of *unknown or non-determined* quantities is much stronger than that of *variables quantities*. This is the result of the fact that their experience with functions has led them generally to solve equations and inequalities (more than 45% of the exercises that appear in their text books and in their class notes asked them to determine the domain of a function) and not to work with activities in which they had to manipulate the notion of variables.

In Table 3 we can see the frequency and the percentages with respect to the total of the pupils from each year and in the whole sample, of the examples of functions proposed by the pupils. The highest percentage, 44.9%, corresponds to examples of similar functions, followed by square functions, 23.2%. We may say that, by showing these examples, the argument used by the pupils in their definitions finds full security (*A function is* $y = x^2 - 3x + 2$): *Definitions find the security needed in what is ostensive*. (Pascal, 1980, p.102).

We should point out that in no case did the pupils determine either the initial set or the final set amongst the ones established by the function that they had proposed as an example, neither did they state its domain. So amongst our pupils we máy highlight a strong presence of laws or criteria that govern the behaviour of the function (**how it varies**) whereas the variable elements (**what varies**) have passed by unnoticed. We must admit that, in general, the pupils presented examples which contained irrelevant properties that are not required by the formal definition of the concept. Therefore there was an *inconsistency* (according to Vinner's meaning of the word, 1990) between the examples proposed and the mathematical definition of the function object, and, in some cases, *inconsistencies* were also presented amongst the examples and their own definitions.

To the question: *Would you give him an exercise or a problem for him to solve? Describe one*. 210 pupils answered yes (65% of the total). Of them, 70% proposed the task of representing functions graphically, mainly similar functions (in these we include linear functions) and square ones. Two of them asked the subject to represent functions that were enormously complex,such as $f(x) = x^3 - 2x\sqrt{x-2}$, this is due to the transparency with which graphic representations of functions are presented, both by the text books and by the teachers in the classroom. It is reduce to a mere algorithm: we give values to the independent variable, we obtain pairs, we place them on Cartesian axes and we immediately join them together to obtain the graph of the function. This leads the pupils to construct knowledge that is too localized, which may be correct within certain limits, but generally the pupils know nothing about the existence of these limits.

Table 3

FREQUENCY AND PERCENTAGE OF PUPILS ACCORDING TO THE TYPES OF FUNCTIONS THAT THEY PROPOSE AS EXAMPLES

	Courses			Total
Types of functions	**2nd BUP** (16 years old)	**3º BUP** (17 years old)	**COU** (18 years old)	
No example	16 11.5	26 30.2	14 14.3	56 17.3
Similar	70 50.3	29 33.7	46 46.9	145 44.9
Square	30 21.6	18 20.9	27 27.6	75 23.2
Rational	7 5.7	11 12.8	4 4.1	22 6.8
In parts	8 5.0	1 1.2	2 2.0	11 3.4
Irrational	0 0.0	0 0.0	1 1.0	1 0.3
Trascendent	2 1.4	1 1.2	1 1.0	4 1.2
Situation of contextualized variabilty	6 4.3	0 0.0	3 3.1	9 2.8
Total of pupils	139 43.0	86 26.6	98 30.3	323 100.0

We should point out that only one pupil (0.5%) proposed an exercise including finding the domain of a function. Here we noticed a great abyss between the activities covered in the classroom, where many calculations are made for the domains of functions (45% of the exercises carried out in the class) and the fact that only one pupil suggested this task. We believe that the phenomenon of rigorous training in algorithms that accompanies, in the present education system, the task of "finding domains" ensures that it is precisely the understanding of the algorithm which replaces the understanding of the meaning of this task: Why and for what purpose is it necessary to determine the domain of a function?

4. CONCLUSIONS

On the basis of everything said earlier, we may now state that, effectively, our pupils manipulate in their definitions terms from three different registers: numerical, algebraic and graphic. If we use Sfard's terminology (1989), we shall say that the pupils develop an *operational* definition -they conceive the function as a certain procedure for calculus-, which is almost synonymous for algorithm. Freudhental (1983) uses an analogous expression to tell us that it involves *operational* definitions, since they only describe the uses that the pupils have recently made of the concepts. According to Salin and Mercier (1988), they would be *constructive* definitions, since they relate the actions that they carry out to construct what for them would be the function object. In this sense, Vinner and Dreyfus state *that pupils pay very little attention to the conceptual aspects of a given notion, whereas they pay much more attention to their computational and operational aspects* (Vinner & Dreyfus, 1989, p.364). Thus, even though the presentation of the notion of function, both in the text books and by the teachers in their classes, is done in its most general and updated way, nonetheless the pupils find and define it only within the limits of the uses they put it to. In this sense, Leonard & Sakur (1990), p.217) assert *that the pupils attempt, in their answers, to reuse the organizations constructed in the class, therefore they contain part of the correct knowledge though very limited by the restrictions inculcated in the educational system.* The *locality of knowledge* is therefore an effect of the didactic transposition (Chevallard, 1991) carried out in the classroom: the *efficiency* to which the pupils are subjected by the restrictions of the educational system as regards their assessment, results in their answers being adapted exclusively to the hierarchy and organization established by the didactic contract in classroom activities. In this sense, we consider that it is very significant that only 4% of the pupils considered a function to be a *transformation or a change between variables*. The phenomena subjected to the change and the cause-effect relationships between variable magnitudes, which were the starting point for the notion of function, are now absent from our classrooms; as a result, the presence of unknown or non-determined quantities is much stronger than that of variables quantities for our set of pupils. We may say that their teaching has misshapen the function object by adapting it so forcefully to their didactic needs, breaking, espistemologically speaking, off from the problems and contexts to which this notion had been linked since its creation. This is a process of "descontextualization" followed by a process of "recontextualization" within the teaching system.

Based on our analysis of the notes taken in the class by the pupils and after studying the text books, we may state that the presentation made of the notion of function which is as general as possible, contrasts with the limitations of the field from which they select the exercises that they offer to the pupils. The restriction of the possibility of assessment to which all knowledge of teaching is subjected results in our teaching system being inflated with exercises involving repetitive applications of algorithmic procedures which are easy to assess: calculus of domains of functions, configurations of value tables, representation of graphs, etc. These tasks conceal all the meaning that the notion of function has regarding the dependency between variables, variability and change, since the algorithmic reduction of mathematical notions has contributed to the blurring of the problem as a power for generating knowledge amongst the pupils and, consequently, has contributed to a loss of the epistemological meaning of these notions: *There is a belief (*amongst teachers) *that knowledge may be taught but that it is up to the pupil to squeeze out the meaning.* (Brousseau, 1987, p.48).

Bearing in mind the set of invariants that the pupils attributed to the function object in their descriptions, which constitute local aspects of their conceptions referring to the intensive component of the concept, we may identify the following typical statements:

- *A function is a certain procedure for algorithmic calculus between numbers*
- *Only relationships that may be described using formulae may be called functions*
- *There is no kind of discrimination between the notion of function and the analytical tools that are sometimes used to describe its law. So the laws and criteria by themselves are considered to be functions, independently of the objects on which they act (initial set, final set, domain)*
- *Every function may be represented on a Cartesian graph using a curve.*

We may state that, in the configuration of these invariants, the working of the present teaching system has a great weight to bear, based to a large extent on the algebraic chart. The restrictions on which this working is supported have been found to be:

-on the epistemological plane: due to the prolonged domination of algebra in the historical development of the notion of function;

-on the didactic plane: due to the strength found by algebra in its refuge in algorithms, strengthened by the restrictions linked to academic assessment (economy of the didactic system).

5. REFERENCES.

BROUSSEAU, G. (1986). *Théorisation des phénomenes d'enseignement des Mathématiques*. Thése de Doctorat d'Etat. Université de Bordeaux.

BROUSSEAU, G. (1987). Les differents rôles du maître. *Actes du XIV Colloque INTER-IREM des PEN. IREM de Nantes*, p. 37-70.

CHEVALLARD, Y. (1991). *La transposition didactique- Du savoir savant au savoir enseigné*. Grenoble: La Pensée Sauvage.

DANE, F. C. (1990). *Research methods*. Berverly Hills, California: Brooks/Cole Publising Company

FREUDENTHAL, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht, Holland: D. Reidel Company.

LABORDE, C. (1988). L'enseignement de la geometrie en tant que terrain d'explotation des phénomenes didactiques. *Recherches en Didactique des Mathématiques*, 9, 3, 337-364.

LEINHARDT, G. , ZALASVSKY, O. & STEIN, M. K. (1990). Functions, graphs and graphing: Task, learning and teaching. *Review of educational Research, 60, 1, 1-64*.

LEONARD, F. & SAKUR, C. (1990). Connaissances locales et triple approche, une metodologie de recherche. *Recherche en Didactique des Mathématiques*, 10, 2.3, 205-240.

RUIZ-HIGUERAS, L. (1994a). *Concepciones de los alumnos de secundaria sobre la noción de función: análisis epistemológico y didáctico*. Tesis Doctoral. Departamento de Didáctica de la Matemática. Universidad de Granada.

RUIZ-HIGUERAS, L. & RODRÍGUEZ FERNÁNDEZ, J. L. (1994b). The role of graphical and algebric representations in the recognition of functions by secondary school pupils. *Proceedings of the Eiteenth International Conference for the Psychology of Mathematics Education*, vol. IV, 137-145.

SALIN, M. H. & MERCIER, A. (1988). L'Analyse a priori. *Actes de l'Universite d'Étè: Didactique des Mathématiques* (p. 203-244). IREM de la Universite de Bordeaux.

SFARD, A. (1989). Transition from operational to structural conception: The notion of function revisited. *Proceeding of the Annual International Conference for Psychology of Mathematics Education* (p. 151-158). Universite Paris.

TALL, D. & BAKAR, D. (1992). Students' mental prototypes for functions and graphs. *International Journal Mathematics Education Science and Technology*, 23, 1, 39-50.

VERGNAUD, G. (1991). La thèorie des champs conceptuels. *Recherches en Didactique des Mathématiques*, 10, 23, 133-170.

VINNER, S. (1983). Concept definition, concept image and the notion de function. *The international Journal of Mathemetics Education in Science and Technology*, 14, 293-305.

VINNER, S. & DREYFUS, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20, 4, 356-366.

VINNER, S. & TALL, D. (1981). Concept image and concept definition in Mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.

NOTE: This work has been carried out in accordance with Research Project PS94-0217, supported by the Directorate General for Scientific and Technical Research (DGICYT), the Spanish Ministry of Education and Science. The authors are members of the Research Group into the Didactics of Science (code 1349) at the University of Jaén, financed by the Education Department of the Autonomous Council of Andalusia (Junta de Andalucia).

THE USE OF ILLUSTRATIONS IN MATHEMATICS TEXTBOOKS

Dora Santos-Bernard

Shell Centre For Mathematical Education, United Kingdom

Abstract

This is part of a research study on the use of illustrations in Mathematics textbooks. Two types of representational illustrations were found to be used in arithmetic problems: relevant and cosmetic. An experimental design with matched groups and using worksheets with different types of illustrations was carried out, collecting the data by individual interviews. It was found that some students considered cosmetic illustrations as relevant and some considered relevant illustrations as merely cosmetic. These findings depended on pupils' mathematical achievement level, the structure of the arithmetic questions, and previous experience in the use of textbooks.

If you look through a current primary Mathematics textbook you can see illustrations everywhere, at least one drawing on every page. Furthermore, if we look at the role each illustration plays in the textbook lesson we can see that it varies from illustration to illustration. So, what is the purpose of including so many illustrations? Evans, Watson and Willows (1987) interviewed representatives of nine major publishing houses in Canada in order to answer the question: Why illustrate textbooks? The main findings were that illustrations were meant to a) dress up the books, b) assist the author to `spin the magic´, c) provide resting points and d) support the text. The designer and editor decided the rhythm, size, place and design of illustrations. Authors said roughly what the illustrations should contain. Authors and illustrators rarely met. Decisions made by editors, publishers and designers were not supported by educational research. Textbook guides emphasise that illustrations are a tool for 'motivation', and there is no mention in regard to the effects illustrations might have in helping or hindering children's understanding.

The use of textbooks is a polemical issue, nevertheless the textbook is one of the teacher's major resources in the classroom (Robitaille and Garden (eds), 1989). Textbooks have illustrations, and illustrations might have an effect on children's understanding and approach to the material presented. Only one of several questions that this research is addressing will be discussed in this paper: how do children use the illustrations presented in their textbooks, specifically in arithmetic problems. The illustrations to be studied are representational, that is drawings and pictures of different scenes that present the context of the problem.

Experimental research done on how children use illustrations in mathematics textbooks is scarce. Poage and Poage (1977) showed that first grade elementary school children make a different interpretation of illustrations than adults. Campbell (1981) suggested that generally children obtain less information from pictures than adults, basing her research on illustrations which showed motion in order to convey the concepts of addition and subtraction. One of her conclusions was that "illustrations may help children understand the concepts of addition and subtraction, but only if the children understand the picture" (p.16), but the question is what is "understanding". Ohlson (1986) analyses and challenges the notion that "illustrations are intended to increase the understanding of arithmetic" (p. 3). She proposes that the function of illustrations is as "objects" in which the child can apply his arithmetic.

Botsmanova (1972) compared the effectiveness of three different forms of illustrations used in problem solving: object-illustrative pictures (which show the objects mentioned in the problem), object-analytical pictures (which illustrate only the essential data) and abstract spatial diagrams. Students from elementary school grades 1 to 4, after solving the problems given to them, were asked to reproduce the illustration. It was assumed that if they could reproduce the illustration after arriving at the solution it meant that the illustration played a definitive role in the problem solving. The object-analytical pictures were reproduced more than the object-illustrative. In another phase of the study with students grades 2 to 4, it was assumed that the time spent solving the problem was related to whether some form of picture was used. Most children in their first approach to the problem did not refer to the illustrations. When re-examining the problem they ignored the illustrative picture but they did come back to the object-analytical ones. These two assumptions underlie this research.

In order to know how children use illustrations in arithmetic problems, for this research a classification of illustrations in such problems was formulated, based on previous classifications made in the area of Science and Reading (Reid, 1990a; Levie & Lentz, 1982) in Mathematics (Botsmanova, 1972; Shuard & Rothery, 1988) and in the area of perception (Goldsmith, 1984; Evans et al., 1987). Considering the relationship between the text and the illustration, illustrations were classified into two groups: **relevant** illustrations and **cosmetic** illustrations.

The relevant illustrations are defined as those which can be used as a source of information in order to answer the arithmetic problem. A relevant illustration can a) be the only source of information, b) present the same information given in the text, or c) have only partial information. The cosmetic illustrations are defined as those with no information necessary for answering the arithmetic problems. The cosmetic illustrations can a) have motivational purposes, or b) portray the context used in the problem.

The aims of this research are to answer the following questions: a) How do children use an illustration that is part of an arithmetic problem?; b) Can illustrations be a cognitive obstacle for children's understanding of an arithmetic problem; c) Can illustrations have a dominant role in preventing students from relating the illustration to the text?; d) From which source do children extract the information to answer an arithmetic problem?; e) Is the way in which the student uses the illustration related to his mathematical achievement level?; f) Is the way in which students use the illustrations related to specific arithmetic questions?

Sample: Seventy eight Mexican children in second grade of elementary school (8 - 9 years old) were selected for the sample. Because the method of obtaining data was through interviews and thus time consuming, children were selected from the same school. They were allocated in three matched groups. The variables considered for the matching system were: mathematics achievement level (scores given from a school standardised test), reading score (given by the teacher), age, gender and teacher. The groups were labelled: group one, group two and group three.

Material used: Three arithmetic problems were selected from current English and Mexican mathematics textbook lessons. Each problem was adapted in order to create three different versions: v1, v2 and v3 The difference among versions was the type of illustration used. For example, in the textbook lesson "The school bus" the problem was to find out how many students could go to the zoo in a bus with eleven double seats. In v1 a cosmetic illustration showing a monkey was used (fig 1). Version two of the same lesson had a cosmetic illustration showing students getting in a bus (fig 2).. Version three showed one part of a bus with double seats and some children in it. The illustration gave partial information, which exemplified the arrangement of the seats inside the bus (fig 3).

Another example is the textbook lesson "The farm". Version one was the original textbook lesson, for which both the illustration and the text were necessary (fig 4). Version two had all the information in the text and the illustration was cosmetic (see fig 5). Version three had the information in the text and in the illustration. It was possible to answer the arihtmetic question only with the text (fig 6).

Design: In total there were nine worksheets, three of each arithmetic problem, and each written in three different ways. Each group was given one worksheet of each topic. For example, group one was given the "The farm" worksheet v1, group two was given the same worksheet but v2, and group three was given v3.

Each child was interviewed individually. He was given the three different worksheets, one at a time. Interviews were used to discover how an individual child on the day of the interview, would use spontaneously the illustrations to answer the arithmetic questions. Also the student was asked to invent another question for each worksheet. In the same manner, he was questioned in order to find out the source of information needed to answer his question. The interviews were audio recorded

Coding system: In order to analyse the data from the transcriptions of the interviews, coding systems were generated. In this paper, only the coding system used to analyse from where the children extracted information will be presented. The sources of information that can be used are: the illustration only; the text only; both the text and the illustration; or previous experience regardless of the problem data. A detailed explanation follows:

Code 1. Assigned to an answer when the child extracts information only from a relevant illustration. It can be said that this is what he is 'expected' to do.

Code 2a. Assigned to an answer when the child tries to use a cosmetic illustration in order to obtain information.

Code 2b. Assigned to an answer when the students uses a relevant illustration but not as 'expected' to be used (i.e. in an 'incorrect' way).

Code 3. Assigned to an answer when the student extracts the information from the text which is the source of information.

Code 4. Assigned to an answer when the child uses the information given in the text, when it was necessary to use the information in the illustration as well.

Code 5. Assigned to an answer when the student uses only part of the information given in the text.

Code 6. Assigned to an answer when the child uses the information in the text as well as the information in the cosmetic illustration.

Code 7. Assigned to an answer when the child uses the information in the text as well as in the relevant illustration.

Code 8. Assigned to an answer when the student's answer is based on his imagination or previous knowledge which does not depend on the information given.

Data analysis: Each question was analysed considering the sources of information used, the reasons given, the worksheet version and the relation it had to the mathematics achievement level of the student. Each group was analysed individually and in relation to the others. In this way patterns can be found for when and why children consider a cosmetic illustration as relevant, read a relevant illustration differently from that expected, or disregard a relevant illustration.

Extracts from interviews: A student with a medium mathematics achievement level, gave the following answers to questions of "The farm" v1 (fig 4). These were coded 2b because he used only the information of illustration but not as expected. (S=student, I=interviewer)

S: "Inside the byre there are 16 cows. How many cows in total?" Seven.
I: How come?
S: There are only seven, these six (pointing at the illustration) and the one that is inside here, you can see her face (inside the byre).
I: But here it says (pointing at the text) inside there are 16.
S: Its a trap, some books are like that!
S: (question 4) "Inside the hen-house there are 6 white chickens and 10 black chickens. How many chickens are there inside the hen-house?" Zero, because you can't see how many there are inside the hen-house.

Another student, with a medium mathematics achievement level, gave the following answer to question six of "The farm" v3 (see fig 6). This question was coded '8' because his answer was based on his ideas about farms.

S: "How many animals are there in total on the farm?" From all they have told me? (Counts the animals on the illustration) Sixty.
I: You counted sixty?

S: No, it's that they are very few so I thought there have to be more animals in the farms. Farms have a lot of animals.

Some Results: The results that are shown in this paper will describe some of the reasons students gave for the use of illustrations on "The farm" and "The school bus" questions. Other variables will not be discussed.

Table one shows the number of students that used or did not use the illustration in questions one, two and five of "The farm". For example, in v1 students were expected to use the illustration. In question one only 14 students did so, and 12 students did not. The main reason given for not using the illustration was that the question had only one number (16) which made the question a statement. They thought the illustration was merely cosmetic. When asked why they used the illustration for the other questions, their reasoning was that each question was independent and had nothing to do with the others. Table two shows that from the 14 students that used the illustration, only half of them used it as expected. The students that did not use the illustration as expected explained that the only existing cows were the ones that could be seen and the ones inside the byre were not to be considered because they could not be seen. The same argument was given for question five.

The children that used the illustrations in v2 and v3 in order to obtain information, explained that because there was only one number (12) it could not be a question. In consequence they referred to the illustration. They did not refer to the illustration in question one because this had two numbers that they could operate on. The same reasoning was used by 13 students with v3 when not using the illustration.

number of students	*Used the illustration*			*Did not use the illustration*		
version	Q1	Q2	Q5	Q1	Q2	Q5
1. necessary illustration	14 of 26	22 of 26	18 of 26	12 of 26	4 of 26	8 of 26
2. cosmetic illustration.	1 of 26	8 of 26	0 of 26	25 of 26	18 of 26	26 of 26
3. optional illustration	2 of 26	8 of 26	0 of 26	24 of 26	18 of 26	26 of 26

Table 1. Source of information used in "The farm" worksheet questions 1, 2 and 5

number of students	*As 'expected'*			*Not as 'expected'*		
version	Q1	Q2	Q5	Q1	Q2	Q5
1. necessary illustration	7 of 14	17 of 22	11 of 18	7 of 14	5 of 22	7 of 18
3. optional illustration	1 of 2	4 of 8	-	1 of 2	4 of 8	-

Table 2. How students used the illustrations in "The farm" worksheet questions 1, 2 and 5

Table three shows the number of students that used or did not use the illustration on "The school bus" worksheets. In v1 and v2 children were not expected to use the illustration to obtain information. Nevertheless almost half of the students with v2 did use the illustration. When asked about the lack of congruence between their answer and what the text said, the majority argued that the text had nothing to do with the questions because it was very clear how many children were getting inside the bus. In v3, from the 19 students that used the illustration in question one only three of them used it as 'expected'. It is noticeable that students did not use the illustration for question two. Their reasoning was that the bus shown in the illustration was for question one. The bus for question two could not be that one because it had to have eleven double seats. When questioned why the first bus did not have eleven double seats, the majority argued that the text could not be used in question one because even if the picture of the bus was not complete (the bus driver and the door were missing) it did had double seats. Few students changed their answer for question one after the discussion.

number of students	*Used the illustration*		*Did not use the illustration*		*Used their imagination*	
version	Q1	Q2	Q1	Q2	Q1	Q2
1. cosmetic-monkey	0 of 26	0 of 26	22 of 26	22 of 26	4 of 26	4 of 26
2. cosmetic-children	12 of 26	10 of 26	13 of 26	14 of 26	1 of 26	2 of 26
3. partial illus-bus	19 of 26	1 of 26	4 of 26	22 of 26	3 of 26	3 of 26

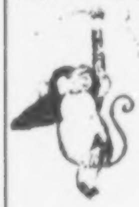
Table 3. Source of information used in "The school bus" worksheet.

Discussion: Students do not necessarily use the illustrations in the worksheets as 'expected'. Several aspects can influence this: a) The decision whether to use an illustration that is part of an arithmetic problem can depend upon the syntax of the arithmetic question being asked, i.e. if the question deals only with one number. b) Some illustrations can have such a powerful effect on the student, that he can disregard the information given in the text in order to answer the arithmetic questions. This dominant effect the illustration has could lead to cognitive obstacles. c) In the same worksheet, students might use the illustration or not depending on the question, thus considering the illustration sometimes relevant and some times cosmetic.

Illustrations might be misleading for some students. Teachers should be aware of this. Authors should be conscious that an illustration might not be read by all students as he might expect them to do. Mathematics educators should be aware of a lack of research done in this topic, and its importance in various areas: textbooks use and design, cognitive obstacles illustrations might create, implications of using illustrations merely as motivational tools. Finally it should not be taken for granted that children read and use illustrations as they are expected to do by teachers and textbook authors.

Reduced Samples of Worksheets

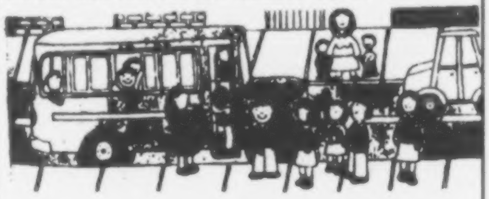
The children at Central City School are going on a day trip to Puebla´s Zoo. The school bus has eleven double seats



1. How many students can Miss Paty take to Puebla´s Zoo?
2. The headmaster of the school got another school bus of the same type. How many children can now go to Puebla´s Zoo?

Figure 1. Version one of the worksheet "The school bus". Cosmetic illustration

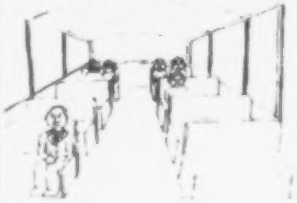
The children at Central City School are going on a day trip to Puebla´s Zoo. The school bus has eleven double seats.



1. How many students can Miss Paty take to Puebla´s Zoo?
2. The headmaster of the school got another school bus of the same type. How many children can now go to Puebla´s Zoo?

Figure 2. Version two of the worksheet "The school bus". Cosmetic illustration

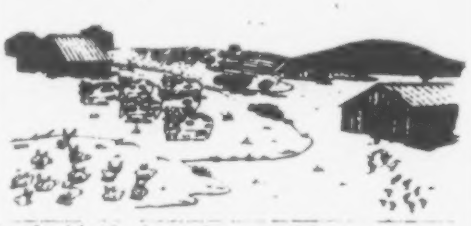
The children at Central City School are going on a day trip to Puebla´s Zoo. The school bus has eleven double seats.



1. How many students can Miss Paty take to Puebla´s Zoo?
2. The headmaster of the school got another school bus of the same type. How many children can now go to Puebla´s Zoo?

Figure 3. Version three of the worksheet "The school bus". Illustration with partial information

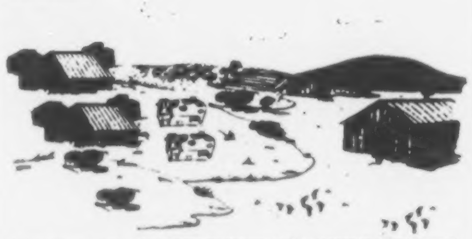
Paco´s Uncle has many animals on his farm. Ana prefers the rabbits; Paco likes the chickens.



1. Inside the byre there are 16 cows. Outside there are six, how many cows in total?
2. There are 12 rabbits. All of them are outside. How many rabbits on the farm?
3. There are 5 white chickens and 9 black chickens outside the hen-house. How many chickens are there outside the hen-house?
4. Inside the hen-house there are 6 white chickens and 10 black chickens. How many chickens in total are there inside the hen-house?
5. Inside the hen-house there are 6 white chickens. Outside the hen-house there are five white chickens. How many white chickens in total?
6. How many animals are there in total on the farm?

Figure 4. Version one of "The Farm". Relevant illustration.

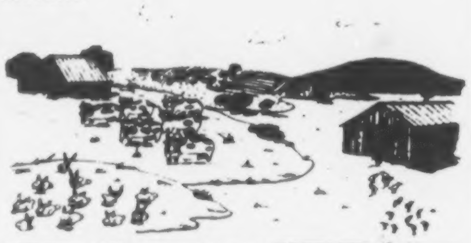
Paco´s Uncle has many animals on his farm. Ana prefers the rabbits; Paco likes the chickens.



1. Inside the byre there are 16 cows. Outside there are six, how many cows in total?
2. There are 12 rabbits. All of them are outside. How many rabbits on the farm?
3. There are 5 white chickens and 9 black chickens outside the hen-house. How many chickens are there outside the hen-house?
4. Inside the hen-house there are 6 white chickens and 10 black chickens. How many chickens in total are there inside the hen-house?
5. Inside the hen-house there are 6 white chickens. Outside the hen-house there are five white chickens. How many white chickens in total?
6. How many animals are there in total on the farm?

Figure 5. Version two of "The Farm". Cosmetic illustration.

Paco´s Uncle has many animals on his farm. Ana prefers the rabbits; Paco likes the chickens.



1. Inside the byre there are 16 cows. How many cows in total?
2. All the rabbits are outside. How many rabbits on the farm?
3. How many white chickens outside the hen-house?

   How many black chickens outside the hen-house?

   How many chickens in total are there outside the hen-house?
4. Inside the hen-house there are 6 white chickens and 10 black chickens. How many chickens are there inside the hen-house?
5. Inside the hen-house there are 6 white chickens. How many white chickens in total?
6. How many animals are there in total on the farm?

Figure 6. Version three of "The Farm". Relevant illustration-optional use.

Bibliography

Botsmanova, M.E. (1972). The forms of pictorial visual aids in arithmetic problem solving. In J. Kilpatrick & I. Wirszup (Eds): *Soviet Studies in the Psychology of Learning and Teaching Mathematics, vol. vi, pp. 105-110.* USA, Chicago.

Campbell, P. (1981). What do children see in textbook pictures? *Arithmetic Teacher, v59, 2, pp. 12-16*

Evans, M., Watson C. & Willows, D. (1987). A naturalistic inquiry into illustrations in instructional textbooks. In H. Houghton & D. Willous (eds): *The Psychology of Illustrations, v2, pp. 86-115.* New York: Springer-Verlag

Goldsmith, E. (1984). *Research into illustration.* Cambridge: Cambridge University Press.

Levie, W. & Lentz, R.(1982). Effects of text illustrations: a review of research. *Educational Communication and Technology Journal, v30, pp. 195-232.*

Ohlson, S. (1986). *Sense and Reference in the Design of Interactive Illustrations for Rational Numbers.* Pittsburg University.

Poage & Poage. (1977). Is one picture worth one thousand words? *Arithemtic Teacher, v24 (May), pp. 408-414*

Reid, D. (1990a): The role of pictures in learning biology: part1, perception and observation. *Journal of Biological Education, v24,* pp.*161-172*

Robitaille, D.F., & Garden, R.A. (Eds.). (1989). *The I.E.A. Study of Mathematics II: context and outcomes of school mathematics.* New York. Pergamon Press.

Shuard, H & Rothery, A. (1988). *Children reading mathematics.* London: John Murray.

EFFECTS OF COMPUTERIZED TOOLS ON PROTOTYPES OF THE FUNCTION CONCEPT

Baruch B. Schwarz, The Hebrew University
Rina Hershkowitz, The Weizmann Institute

It has been shown in several studies that linear functions are prototypes of functions. In the present research, two groups that underwent different rich educational programs on functions are contrasted on prototypical knowledge that developed. The first group (G1) underwent a rich curriculum, but tools and activities were quite "ordinary". The second group (G2) followed a curriculum based on computerized tools and open-ended activities. It is shown that, while G2 students often prefer to use linear functions to exemplify properties or to solve problems when it is possible, other functions are invoked, when they are needed: quadratic, power, piece-wise linear, or atypical functions. Families of functions are also often used. In contrast, G1 students used extensively linear functions (even when their use led to aberrations), referred almost exclusively to atypical functions whenever richness of the concept needed to be expressed, and used less examples than G2. Moreover, the justifications given by G1 students were less articulated and principled than those of G2 students.

PROTOTYPES AND CONCEPT LEARNING

Prototypical examples are important for the development of concepts. For example, properties of fruits are better learned through the prototype "apple", than through olives. However, as has been shown in geometry, if prototypes are persistently too dominant, they impede learning, because they are used as frame of reference in the judgment of other examples. In this case, the attributes of the prototypes are imposed on other examples of the concept (Hershkowitz, 1989).

The concept of function uncovers a similar phenomenon. Karplus (1979) showed that most high school students use linear inter/extrapolations to find values of functions whose graph passes through two given points. Karplus used Item I3 in the appendix (what he called the "bacteria puzzle") to show this result. With Item I3c where the linear extrapolation leads to an aberration (a negative number of bacteria), Karplus showed that a non-negligible part of high-school students were still linear. Markovitz, Eylon, & Bruckheimer (1986) asked some of Karplus' items, and added formal items to find that many Grade 9 students declared that the linear function only passes through two given points. Similarly, a significant part of the students thought that no graph can pass through three non-collinear points. Therefore, these studies show the prevalence of the linear function as an almost exclusive prototypical example.

TWO CURRICULA FOR FOSTERING THE FUNCTION CONCEPT

Many new curricula were implemented to foster the concept of function. They are often sensible to the fact that students need to be exposed to many examples, and families of functions. There is a pedagogical belief that computers can help in this endeavor. Experimental studies showed that this belief has psychological roots (Schwarz & Dreyfus, 1995).

Two curricula on functions were recently developed and implemented in Israel by the Weizmann Institute. Both are based on the same syllabus, and consist of a one year long course for Grade 9. The first curriculum that was developed about 7 years ago, is traditional in the sense that (a) subconcepts, representations, and procedures are presented and trained in a linear way, the formal definition of function and its subconcepts being introduced quite early; (b) students are encouraged to work in an exploration mode, but their exploration is coached by the textbook and/or the teacher, the main goals being hidden from the learner.

The second curriculum that was developed in the last 3 years, is based on (a) team work on open-ended problems, the majority of which being problem-situations; (b) the extensive use of technological tools; (c) a "concurrent" teaching approach (as opposed to the linear way above), subconcepts and procedures being reinvented by the problem solver on the basis of intuitive knowledge (d) fostering mathematical written and oral discourse. It is important to note that the computerized environment did not function as a tool only but as a facilitator of mathematical activities such as modeling (see for example Hershkowitz & Schwarz, 1995).

The two curricula presented students to items similar to I1 and I3. These items were given to exemplify the richness of the concept of function before students systematically studied linear, quadratic or other functions.

The present study compares between two groups of students (G1 and G2) who followed the two curricula. Many differences can be found between them: different teachers, tools, kinds of interaction, and activities. However, far from evaluating the curricula, the results are analyzed to differentiate prototypical knowledge from a rich representation of the function concept.

An important theoretical reminder relating to the learning of the concept of function with computerized environments will be useful for the following. In a previous study, Schwarz and Dreyfus (1995) showed that two computer actions particularly contribute to conceptual learning: the passage to another representation, and the manipulation of what is called representatives. Representatives are partial embodiments of graphs, tables or formulae that are displayed by computers. For example, two concretizations of a graph differing by their scales only (and not by the function they represent) are different graphical representatives of the same function. Students who manipulate representatives were able to interpret them to infer properties of functions.

THE RESEARCH QUESTIONS AND HYPOTHESES

We planned to characterize the richness of the concept image of the two groups. The research questions that will be discussed in this report are: (1) How rich is the concept image of the two groups, as expressed in the examples and justifications produced? (2) In particular, what is the status of the linear function?

Concerning the first question, we hypothesized that, although the two groups were both exposed to a rich spectrum of functions, G2 would be more inclined to retrieve and use different kinds of functions in order to solve problems. As for justifications, it checks whether judgment on functions is driven by principles rather than on visual features. We hypothesize that G2 would prevail on G1 for providing justifications based on principles. Such an hypothesis, if checked, will indicate whether examples used by students are prototypical, or whether they exemplify properties.

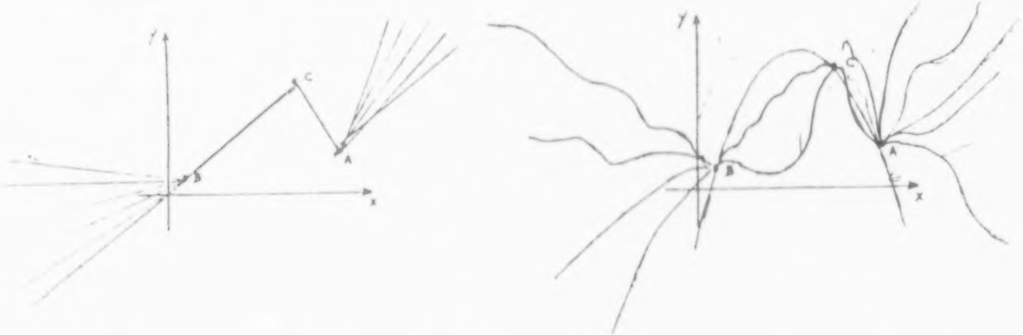
The second question whether linear functions are prototypical examples. We hypothesize that both groups know that linear functions are not the only ones whose graph passes through two given points, as opposed to the results obtained by Karplus (1979), and Markovits et al. (1988). However, there are still other questions that relate to the prevalence of the linear function: Is the linear function preferred over other functions when it is legitimate to use it? Do students use linear interpolations or extrapolations when there is no cue to such a use, even when it leads to aberrations? All these sub-questions of the first research question are quite open, concerning differences between groups. Moreover, this question leads to investigate the possible existence of other prototypical examples.

METHODOLOGY

Two groups of Grade 9 students participated in the study. Both groups belong to the higher 20% of the general population in Israel. G1 (n=71) underwent the first curriculum on functions; G2 (n=32) underwent the computer-based curriculum. A questionnaire was constructed to investigate the research questions. It was administered to G1 and G2 at the very beginning of Grade 10. The questionnaire includes eight items (some of them appear in the appendix).The three first items are about functions whose graph passes through two (Item I1 in the appendix), or three given points (Item I2), and about the influence of a referent (here the "Bacteria puzzle") on the answers (Item I3 in the appendix). The results on these items are documented (Karplus, 1979; Markovitz, Eylon, and Bruckheimer, 1986). They showed the same tendency: students are generally "linear", meaning that the linear function is often the only one that passes through two given points, meaning also that linear interpolation and extrapolation were preferred, even when they led to aberrations (negative numbers of bacteria on Item I3c). The present report focuses on the analysis of the results for these three items only.

RESULTS

To answer the first research question, we first analyzed the kinds and number of examples used in the questionnaire. The number of examples (appearing in graphs) was higher in G2 than in G1: G1 students gave 1.13 examples on I1 and 1.17 examples on I2, in contrast with 1.31 and 1.56 examples for G1. Similarly, G1 hardly used families of functions (0.10 per questionnaire), in contrast with G2 (0.78). For example, the following figure shows two families displayed by G2-students.



The first one shows a family of piece-wise linear functions. The second one contains a quadratic function along with a family of atypical functions, that is, graphs displaying a freehand line passing through the given points.

Therefore, G2 uses more examples, and more families of functions to solve problems. Analyzing the kinds of examples invoked in I1, I2, and I3 gives more information about the richness of the concept image of the two groups. I1a shows that G1 as well as G2 were very "linear": many students of both groups chose the linear function as the only example of function passing through two given points (59% for G1, 53% for G2). However, none of G2 students answered that the linear function is the only one, where 17% of the G1 students who chose the linear function believed so.

The following data interestingly completes the picture: 36% of G1 students, and 47% of G2 students drew another function. Moreover, observation of these results shows that most of G1 students who drew nonlinear functions on I1a, drew "atypical" graphs. In fact, 11% (resp. 25%) of G1 (resp. of G2) used functions that were not atypical, or not exclusively linear, for I1a. Similarly, among the 80 graphs drawn by G1 (1.13 per student), 62% where linear, 5% parabolic, 6% piece-wise linear, 21% atypical, and 5% did not answer. In contrast, among all the 42 graphs drawn by G2 (1.31 per student), 55% were linear, 25% parabolic, 5% piece-wise linear, and 17% atypical.

In summary, while many students of the two groups used linear functions to exemplify one function passing through two given points, all students in G2 (as opposed to G1) knew that this was only one example. Moreover, G1 students used almost exclusively linear or atypical functions, G2 often used other types of functions.

The justifications obtained by the two groups on the items of the questionnaire were very different, and gave also precious indications on the concept image of the two groups. For example, the justification written by a G2 student for the first graph shown above was:

> *It can be one graph in the domain, and it varies in infinite ways outside. Also, it can change in the domain if I wish so (but only if there are not two images for one preimage). See my drawing!*

Typically, the justifications given by G1 students are shorter, such as sentences of the kind: *An infinity of graphs passes through three points.* To differentiate between justifications, we introduced the term "idea unit". The first justification contains three idea units: (1) it varies in infinite ways outside; (2) it can also change within the domain: (3) there are not two images for one preimage. The second justification contains one idea unit. The length of this report cannot enable us to discuss the methodological problems (such as validation) raised by this notion. However, equipped with this tool (presented here intuitively), we could observe that G2 wrote significantly more idea units than G1 on all the items of the questionnaire. For example, G1 students wrote 1.34 idea units in average for I2, in contrast with the 2.16 of G2 students. This data indicates that G2 students, in their justifications, lean much more than G1 students on principles.

In summary, G2 students gave more articulated justifications and more examples. This confirms our first hypothesis, that is that G2 students have richer concept image.

The results on I3 go further and clarify the status of the linear function so frequently chosen in I1 by the two groups. Among G1 students who chose only a linear function on I1a, 48% did a linear interpolation in I3a, 43% a linear extrapolation on I3b, and 43% a linear extrapolation in I3c, the respective percentages for G2 being 35%, 35%, and 18%. All students who did a linear extrapolation for I3c reached a negative number of bacteria. Therefore an important part of G1 students use linear extrapolation when it is totally inadequate. In contrast, G2 students use linear properties or strategies in a flexible way, always considering whether they match the problem under consideration. For example, a typical G2 answer to I3a was that of S22:

> *"You can say that the number of bacteria will be close to three thousand because at 10 °C they were five and at 25 °C they are two, so when the temperature rises, their number diminishes, so it will be between five and two. At 20 °C, they are close to three (between 5 and 2)".*

But then, the student drew an atypical curve and added:

> *"All the answers are based on the assumption that when the temperature rIses, the number of bacteria diminishes. But maybe it's a special kind of bacteria without any proportional link between temperature and number of bacteria. (For example at 10 °C they are five, at 15 °C their number rises to 10, and at 25 °C there are only 2 left)."*

In this example, the student clearly states that linear extrapolation is based on an assumption she thought reasonable, although she is aware that it may be false. And indeed, S22, like most of G2 students does not extrapolate on I3a.

In summary, many from G1 students often choose linear functions because they are prototypes from which they may infer aberrations such as the extrapolation leading to a negative number of bacteria. On the contrary, G2 students use linear strategies or properties in a flexible way, depending on the conditions under which the problem is posed. In I1, there is no reason not to use a linear function to exemplify a functional property. In I3c where a linear function is inadequate, the student searches for another kind of function, an atypical function, that matches the constraints of the problem. For G2 students, linear inter/extrapolation (learned on linear functions) is a legitimate strategy applicable to other functions, when this application is reasonable. The strategy of linear inter/extrapolation, when used, does not imply that the function under consideration is linear.

CONCLUSIONS

We showed that G2 students had a rich concept image. They used a broad spectrum of functions to reason with, and evoked them to exemplify properties or strategies. Linear functions were used frequently due to their usefulness. The richness of their concept image was also expressed by well articulated justifications. In contrast, G1 students used linear functions as prototypes: they were the functions that pass through two given points. Linear extrapolations were not only reasonable but necessary.

REFERENCES

Hershkowitz, R. (1989): Visualization in geometry - Two sides of the coin. Focus on Learning Problems in Mathematics, 11, 61-76.

Hershkowitz, R., & Schwarz, B. B. (1995): Reflective processes in a technology-based mathematics classroom. Paper presented at the AERA conference, San Francisco.

Karplus, R. (1979): Continuous functions: student's viewpoints. European Journal of Science Education, 1, 397-415.

Markowitz, Z., Eylon, B., & Bruckheimer, M. (1986): Functions today and yesterday. Focus on Learning Mathematics, 6(3), 18-24.

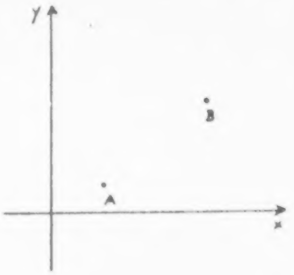
Schwarz, B. B., & Dreyfus T. (1995): New actions upon old objects: a new ontological perspective on functions. Educational Studies in Mathematics, 29(3), 259-291.

APPENDIX

11a. In the given coordinate system draw the graph of a function such that the coordinates of points A and B represent the preimage and the corresponding image of the function.

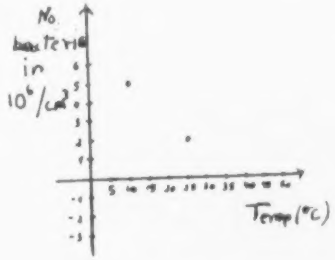
11b. The number of different such functions that can be drawn is: (a) 0 ; (b) 1; (c) 2 ; (d) more than 2 but less than 10 ; (e) more than 10 but not infinite; (f) infinite.

Explain your answer.



13 A scientist undertakes a study on bacteria cultures. Bacteria species are known to live at a different range of temperatures, therefore the number of bacteria in such a culture is dependent on temperature.

The number of bacteria at the temperatures of 10°C and 25°C is marked in the following coordinate system.



a) At the next step of the study, the scientist needs to know the number of bacteria at 20C°. What can you say about the number of bacteria at that temperature? Explain your answer.

b) The scientist wants to predict the number of bacteria at 30°C. What can you say about the number of bacteria at that temperature?

Explain your answer.

c) The scientist wants also to predict the number of bacteria at a temperature of 45°C. What can you say about the number of bacteria at that temperature? Explain your answer.

AFFECTIVE CONSTRAINTS IN MATHEMATICS EDUCATION

Beth Southwell and **Mon Khamis**
University of Western Sydney, Nepean, NSW, Australia

Fear of failure or lack of confidence, as well as some contextual factors inhibit understanding and enjoyment in mathematics. The way in which students perceive mathematics and learning mathematics has an impact on their success in the subject. Following a smaller scale survey of beliefs concerning mathematics held by primary and secondary students and teachers, this study looks in more detail at a larger sample of over 2000 secondary school students, makes further comparisons and suggests trends in beliefs of students at secondary school level. This current study confirms most of the findings from the previous study.

***

Too many numbers,
Too many signs,
Too confusing for the mind!
Too much to think about.
Too much to know.
And it is boring, 'cause I say so!

- Rosemary, Year 7

Is this how all high school students see mathematics? How is Rosemary being affected by her view of mathematics?

Certainly, her poem implies that she does not find any enjoyment or success in mathematics. One can only surmise possible reasons for this but it is alarming to think that she has such negative views of the subject that she is willing to write them in a poem to which others had access. Nor was she alone in her class. Approximately half the class wrote poetry in a similar vein.

Rosemary's poem implies several beliefs about mathematics as she experiences it - and, after all, that is the way that everyone develops his or her beliefs about mathematics. From her emphasis on numbers, it seems reasonable to assume that Rosemary believes that mathematics is just arithmetic. This seems to be a fairly common belief among school students at all levels. The fact that so many people, including the general community, appear to believe that mathematics is just arithmetic could be the result of several factors. These include the past lack of geometry and other branches of mathematics in school syllabuses, particularly in the primary syllabus. Or it could be the result of the emphasis placed on numerical computations in public statements and tests such as the Basic Skills Tests. Whatever the cause, such a belief not only downgrades other branches of mathematics, but it also inhibits students from gaining rich experiences which are related, sometimes more directly than arithmetic, to their everyday life.

Rosemary's emphasis on signs while again indicating a belief that mathematics is arithmetic, also indicates a belief that mathematics is highly symbolic. Of course, this is so. The indication is, however, that Rosemary sees that symbolism only in relation to arithmetic and does not appreciate the richness, beauty and usefulness of mathematical symbolism in general.

The reference to mathematics being confusing could refer to the 'signs' or it could be a general statement indicating a feeling of helplessness in relation to the subject. Either way, the damage which has been done to Rosemary's perception of herself as a mathematician is obvious. It is all 'too much'! She may as well have said, "I can't think when I see all these signs. I can't learn it." How little confidence she has in herself and in her ability! How destructive of her self-image!

The final bold, almost defiant statement about boredom could be seen as an attempt to justify her previous comments. This implies a certain feeling of guilt or even rebellion, against anyone who might have counter beliefs to hers. It does emphasise the extremely personal nature of one's engagement with mathematics, or any subject for that matter. It seems, however, that mathematics arouses more polarised feelings and beliefs that most other subjects.

As has been indicated, Rosemary is not alone in the feelings of helplessness, frustration and boredom which she has expressed in her poem. Rosemary in her time will leave school, take up a vocation, perhaps marry and have children and become part of the general community without ever having the opportunity to experience mathematics as a relevant, interesting, exciting subject. Perhaps this is why the general community of the present appears to have certain beliefs about mathematics and certain expectations as to what is taught in schools and what students should know as a result. The effect of these community beliefs appears to be that unrealistic and probably limiting expectations are placed on students and teachers, schools and school systems in general.

The first contact schools have with the community at large is through the parents of the children in the school. Consequently, some indication of parents' beliefs about mathematics and about their children's mathematics could provide possible trends. Once such trends are established, there may be strategies that can be employed to gradually change the beliefs of at least a few of the members of the general community towards more helpful ones.

Previous small scale research carried out by the writers (Southwell & Khamis, 1991) using a sample of 510 primary and secondary students and primary teachers indicated that the following beliefs were held:

1. You are either good at mathematics or not.
2. Answers in mathematics are either right or wrong.
3. If you do not get the right answer, you just start again.

4. Mathematics is important for every day life.
5. Memorising facts and procedures is the way to learn mathematics.
6. Mathematics is arithmetic.

In this previous study, the secondary section of the sample (310 subjects) were all female. The question arises, then, as to whether male students have different beliefs about mathematics and about themselves as mathematicians. Also how are they affected by the beliefs their parents hold? These formed the basis of the research questions for the current study. They were:

1. Do male secondary school students hold the same beliefs about mathematics as their female counterparts?
2. Do male secondary school students see themselves as mathematicians in the same way as female secondary students do?
3. Do secondary school students have perceptions of their parents' beliefs about mathematics which are different from their own?

The Sample

As the secondary section of the sample in the previous study was totally female, it was felt that a sample of both male and female respondents may provide more specific information. Consequently the secondary survey instrument was administered to a larger sample from six schools consisting of both male and female as indicated in Table 1.

School	Female Students	Male Students	Total
School 1	175	219	395
School 2	243	246	493
School 3	193		193
School 4	355		355
School 5		367	367
School 6	183	158	344
TOTAL	1149	990	2129

Table 1. Distribution of Sample

The Survey Instrument

The survey instrument used consisted of items adapted from Schoenfeld(1989) and Way (1990). Information was sought on secondary school student' beliefs concerning:

(i) their mathematical success or failure;
(ii) the nature of the mathematics learned;
(iii) the learning of mathematics in relation to other subjects;

(iv) learning geometry; and
(v) perceptions of parental expectations.

The subjects were asked to respond to each item on a four part Likert scale ranging from "1= very true" to "4 = not at all true". As well a number of open-ended questions gave students the opportunity to express their views without constraints.

The Analysis

The analysis of the responses of the 2147 subjects was carried out by finding means, standard deviations and significance levels for the responses made by female and male subjects. In general, the analysis followed similar trends as in the previous study. There were, however, differences of interest. These differences are reflected in the difference in the male and female responses in the current study, since all secondary school respondents in the earlier study were female.

A t-test was used to compare mean responses for females and males.

Results

Significant differences ($p < .001$) occurred in the responses concerning reasons for getting good grades. These are shown in Table 2. The difference between female and male students is significant ($p<.001$) on all five reasons with the highest response being the perception that good grades are achieved because of the teacher's liking of the student. Allied with this is the significant ($p<.001$) result that the main reason for trying to learn mathematics is 'to make the teach think I'm a good student'.

	Totals N = 2147		Females N = 1158	Males N = 974	Signif Level
	Mean	S.D.	Mean	Mean	
1. It's because I worked hard.	1.5	.6	1.5	1.6	p < .001
2. It's because the teacher likes me.	3.6	.6	3.6	3.5	p < .00
3. It's just a matter of luck.	2.8	.8	2.7	2.9	p < .00
4. It's because I'm always good at mathematics.	2.6	.8	2.7	2.5	p < .00
5. I never knew how it happened.	3.1	.8	3.1	3.2	p < .00

Table 2. Students' Beliefs About Why They Get Good Grades in Mathematics

The reasons for getting poor grades were not so divided with the only significant results being the responses "Because the teacher doesn't like me" and "Because I'm just not good at mathematics". The differences between male and female responses are given in Table 3. This shows the high response again concerning the teacher's liking or dislike of the student being a critical factor.

	Totals N = 2147		Females N = 1158	Males N = 974	Signif Level
	Mean	S.D.	Mean	Mean	
1. It's because I didn't study hard enough.	1.6	.7	1.6	1.6	-
2. It's because the teacher doesn't like me.	3.5	.7	3.6	3.4	p < .00
3. It's just bad luck.	2.9	.8	2.9	2.9	-
4. It's because I'm just not good at mathematics.	2.8	.9	2.7	2.9	p < .00
5. It's because of careless mistakes.	1.7	.7	1.7	1.8	p < .05

Table 3. Students' Beliefs About Why They Get Bad Grades in Mathematics

No great differences were indicated in the female and male students' beliefs about the nature of the mathematics they learn. There is a difference, however, in the responses of the females in the current study and those in the previous one. The current respondents are not as convinced as the previous one that mathematics is thought provoking.

While both groups of students feel that "good mathematics teachers show students the exact way to answer mathematics questions you'll be tested on", their responses were significantly different (p< .001). As well, the female respondents were more convinced of this than those in the previous study.

Female respondents believe that "some people are good at science and some just aren't". Their response is significantly different from the males (p< .01). Despite this, the difference between female and male respondents' belief that good science teachers show students the exact way to answer questions is not as significant (p< .05).

The t-test indicates that there is a significant difference (p<.01) between the degree to which females and males believe that 'everything important about mathematics is known already by mathematicians'. This result is apparently inconsistent with the another significant difference between females and males. This is that things can be discovered about geometry without being taught (p< .001). This latter result appears to be supported by the difference between female and male students on the degree to which 'you can be creative and discover things for yourself" (p<.05). In contrast, both female and male respondents believe that they can only verify something a mathematician has already shown to be true.

Although both females and males believe the best way to do well in mathematics is to memorise all the formulas, the difference between females and males is significant ($p< .01$). This is consistent with the difference ($p< .001$) between females and males in beliefs about geometrical constructions and the necessity to memorise the way to do them. Learning geometry as a means of better understanding mathematical thinking differs for female and male students ($p< >01$).

There is a significant difference between females and males in the way in which they respond to getting wrong answers. While both tend to 'start all over in order to do it correctly', females seem to do this more readily than males.

Perceptions of students about their own mathematical ability and about their parents' expectations for them differ between female and male students in several aspects. Females see themselves as being average students more than males do ($P<.01$). Females tend to complete their homework more often than males ($p< .001$). Males believe it is more important to do well in mathematics than females do($p<.01$). Both mother's and father's perceptions of the importance of mathematics are seen to be greater by males ($p<.01$).

Some items in the survey which invite some attention are some in which there is no significant difference between female and male respondents and some which reveal a high response. In both categories are getting bad grades because of not studying hard enough or because it is just bad luck, the belief that mathematics is mostly facts and procedures that have to be memorised (significant at .05 level), the belief that some people are good at mathematics and some are not, that in mathematics something is either right or wrong. This last belief is also held for science though not for English. Other beliefs held are that mathematical problem solving is important for everyday life, that mathematical thinking is what we do in solving problems, and working in groups is helpful. At the same time both females and males report that they do not often work in groups. Both females and males, however, appear to want to do well in mathematics.

Discussion

The reasons given by female and male students for getting good and bad grades follow the generally accepted view that females attribute their success to working hard and luck while males attribute their good results to being good at mathematics as well as working hard. The interesting aspect of these current results is the significant role played by teachers in the students' perception of their success and failure. They believe they get good or poor results because the teacher either likes them or not and they want to learn mathematics so the teacher will consider them a good student. There is no mention of the content being learnt and taught and nothing about the quality of the teaching. It seems to be the personality of the teacher which is being referred to. Such a major influence played by the teacher could be due to several reasons. These include the low perceptions which the students have of their own ability which makes them see the teacher as being the holder and dispenser of all wisdom in relation to mathematics. This is supported by other beliefs held that mathematics is basically facts and procedures which have to be memorised. The teacher is the assessor then to determine the degree to which these facts and procedures have been memorised. This also links with the belief that good mathematics teachers will show students the exact way to answer mathematics questions they will tested on.

While no significant gender differences were observed in the respondents beliefs about mathematics itself, the trend was to see mathematics as mostly facts and procedures that have to be memorised. This was emphasised also in relation to geometrical constructions.

The beliefs that mathematics is a subject in which there is nothing more to learn, a subject in which you either get the right answer or you are wrong and that you are either good at mathematics or you are not indicate a very limited view of mathematics and mathematicians. This could be because school syllabuses are very tightly structured and the demands of public examinations inhibit the methods and content introduced by teachers.

In contrast to beliefs about ability to do mathematics and science is that concerning English. This supports the view that these subjects are seen as different kinds of subjects to English.

Again, the emphasis seems to be on being right or wrong. When one gets a wrong answer, the tendency is to start over again and not to use the work already done. As has been shown in computer programming, the process of de-bugging is a very useful one in helping the programmer become more skilled. This has not carried over into other areas of education. This is true in mathematics possibly because of the limited view the respondents have of mathematics as a discipline and also because of their own lack of confidence in themselves as mathematicians.

The result that the perception of both female and male students that their mothers consider mathematics important less than their fathers seems to indicate that the community in general has not moved very far towards equal opportunity for all students.

Conclusion

This preliminary analysis of survey responses from over two thousand students has highlighted some issues which support previous findings. It also raises some issues, such as that of the role of the teacher which will need to be investigated more thoroughly. For further exploration, also, is the question of whether there are any significant differences between the responses of the females in all girls schools and those of females in co-educational schools. A similar question arises in relation to male students in all boys schools and those in co-educational schools. The one heartening point in contrast to Rosemary's poem is that the respondents do want to learn mathematics. It is up to teachers to ensure that the mathematics they learn is enjoyable, exciting, useful and challenging.

References

Australian Education Council National Statement on Mathematics for Australian Schools. Canberra: Curriculum Corporation.

Schoenfeld, A., (1989) "Explorations of Students' Mathematical Beliefs and Behaviour." *Journal for Research in Mathematics Education*. Vol 20, No 4, July, pp 338-355.

Southwell, B. & Khamis, M. (1991). 'Affective Considerations in Assessing Mathematics", in Izard, J.(Ed.).(1991). *Assessment in the mathematical sciences*. Melbourne: ACER.

Way, J. (1990). *Group work: Teachers' attitudes and practices*. Unpublished MEd project.

On the Ability of High-School Students to Cope with a Self-Learning Task in Algebra

Jonathan Stupp and Shlomo Vinner

Dept. of Science Teaching, Hebrew University, Jerusalem

Abstract: *This paper describes an experiment conducted with 10th grade students, and pre-service teachers who were given a self-learning task in which they had to characterize, along guiding steps, a necessary and sufficient condition on the coefficients of a quadratic equation, which will insure that the roots of this equation will be in the interval (-1,1). In this experiment we wanted to check the ability of students to connect facts they already knew and to draw the necessary conclusions. We wanted also to see how students can pass from graphic aspects to algebraic ones and vice versa. Some additional interesting cognitive issues turned up during the students' performance.*

Introduction: In recent years there has arisen the need for alternative ways to assess students' mathematical ability, as conventional tests seem to fail in bringing out the more creative skills of students. In our research, we tried to assess students' mathematical ability by testing their performance working on a self-learning task, given to them as a collection of guiding steps which were supposed to lead the students to discover some new facts about a concept which was already known to them. In this case, the characterization of the coefficients of a quadratic equation which is both a sufficient and a necessary condition for the roots to lie in the interval (-1,1). We had mainly three goals in mind:

a) Testing students for their ability to follow along a collection of guiding steps, to relate the relevant steps and eventually draw the required conclusion. In order to perform well, students have to show a "relational understanding" according to Skemp (Skemp 1979), which is a higher level of understanding than the "instrumental understanding" usually required from students. Schoenfeld (1987) describes a situation in which about 30% of students could not solve a problem which was almost an immediate conclusion of a theorem they had proved just before. Students usually withdraw from even trying to solve a problem they have never seen before even if it is rather easy. In our task, we tried to show the students that it is not as difficult as it seems, and that they should enjoy the challenge (which at least some of them eventually did).

(b) Since several times during the task the students had to draw algebraic conclusions using a graph, we could check their ability to perform the transition from algebraic aspects to graphical aspects and vice-versa. Arguing correctly about algebraic facts using graphs, shows

c) Since in self-learning tasks, the student have to interpret their way of thinking and reflect upon it, interesting cognitive issues turn up, issues which do not become explicit during routine learning. For example, in the present task the first step whose purpose for us was only a technical one (to reduce the problem to the case of a positive coefficient of x^2) brought up some interesting aspects of students' dealing with symbols.

Methodology The task was given to 61 tenth graders at the School for Science and Arts in Jerusalem. This is a boarding school for rather gifted students in the fields of science and art. The task was given to students who do not excel in mathematics, but nevertheless possess an above average level of mathematics abilities relative to the general average level in Israel. In addition, the task was given also to 12 prospective mathematics teachers, studying at a teachers' college.

All the students possessed the necessary pre-requisite knowledge of investigating the roots of a quadratic equation and the graph of a parabola.

The text of the task was constructed in such a way that steps 1-4 below led the students toward the three inequalities which constitute the characterization. At the end of step 4 the students can conclude that if the roots of the equation lie in the interval (-1,1) then these inequalities must hold, and therefore they form a necessary condition. The students were already familiar with the term "a necessary and sufficient condition". Steps 5-7 below guide the students to show that the condition is also sufficient. The last step checks whether the students know how to apply in a particular case the criterion they have just established.

The students were given ninety minutes and were told that they may consult the teacher whenever they encounter a difficulty which prevents them from going on. They made use of it only at the beginning but after a slight "push" most of them worked by themselves. Although the students were told they would not be graded, they showed high motivation to complete the task.

The Working Sheet

Text of the Task:

Given the general equation of a parabola: $f(x) = ax^2 + bx + c$, $a \neq 0$, the goal of the task is finding necessary and sufficient conditions for the parabola to have roots in the interval (-1,1). Please explain every step you make.

1) Suppose $a < 0$. Write, in terms of a,b,c another parabola having the same roots as the given parabola, but with a positive coefficient of x^2.

Therefore, we can assume from now on without loss of generality that $a > 0$.

2) Draw a graph of a parabola with two roots in the interval $(-1,1)$, and a parabola with only a single root which lies in this interval. Which of the following situations exists? a) $f(1)>0$, $f(-1)<0$; b) $f(1)<0$, $f(-1)>0$; c) $f(1)<0$, $f(-1)<0$; d) $f(1)>0$, $f(-1)>0$.

3) Express the two inequalities which you pointed out in terms of a,b,c.

4) In both parabolas of part 2 the vertices of the parabolas lie in the interval $(-1,1)$. Why? Which of the following inequalities expresses this fact? (remember that $a>0$).

a) $-b > 2a$; b) $|b| < 2a$; c) $-2a<b$; d) $|2a|<b$

From what you have shown so far you can conclude that if the roots of a parabola lie in the interval $(-1,1)$ then the three inequalities of (3) and (4) which you have pointed out must hold. Hence these three inequalities form a necessary condition for this fact.

Now you have to show that these inequalities are also sufficient, which means that you have to show that if these inequalities hold, and the parabola has real roots they must lie in $(-1,1)$.

5) Assume that the parabola has only a single root and satisfies the 3 inequalities. Using the graphic meaning of a single root, from which of the inequalities can you conclude that this root lies in $(-1,1)$?

6) Assume the parabola has two roots but not both of them lie in $(-1,1)$. Show that there are 5 distinct cases and draw a suitable parabola for each case.

7) Show that for each of the 5 cases in (6) at least one of the 3 inequalities is violated. Explain that if the 3 inequalities hold, the parabola cannot have roots outside $(-1,1)$.

8) Given the parabola: $f(x) = 5x^2 + (k+3)x + k - 1.8$

a) What is the range of k for which the parabola has two distinct roots which lie in $(-1,1)$?

b) For what values of k the parabola has only a single root which lies in this interval?

Analysis of the Students' Performance:

We will analyze each step of the task separately. We will compare the quality of the performance between the school students and the prospective teachers. In each step the answers will be classified from the cognitive point of view into three categories. The first category will include the answers which show an ability to work correctly with formal algebraic symbols and a "relational understanding", namely an ability to connect several facts and draw the necessary conclusion.

Answers in the second category will show a correct intuitive understanding but without the ability to argue mathematically correct. The third category will include answers which are wrong and show that the student did not understand the relevant point. These will also include answers which use "pseudo analytical" arguments, by which we mean answers using seemingly relevant facts which have nothing to do with the required conclusion (Vinner 1994). The meaning of the categories may differ slightly in different steps, but in any case the categories define different levels of thinking.

Step 1: The role of this step was to reduce the problem to an equation with a positive coefficient of x^2, and the expected answer was of course to multiply the equation by -1. This "innocent looking" step produced some interesting results.

Category I (twenty-four students = 20 + 4, the underlined number is the number of the prospective teachers). Here we included the expected answers with a correct argument. Most students argued that the multiplication by -1 does not change the roots and wrote the new equation: $-ax^2 -bx -c = 0$. There were also 4 correct graphic arguments. The students showed that multiplication by -1 does not change the intersection of the graph with the X-axis, and produced an appropriate drawing.

Category II (thirty-five students = 30 + 5) Most of the answers in this category were technically correct answers as regarding the multiplication by -1 but without any argumentation. There were also some answers with correct arguments which were included in this category. The students showed that if we multiply a quadratic equation by -1 the formula which we use to find the roots remains invariant. These students paid no attention to the fact that we get an equivalent equation whenever we multiply by -1 both sides of an equation and it has nothing to do with the way we solve the equation. Instead of thinking about the meaning of the "root", the students think about the procedure to find it. The procedure becomes a substitute for the concept (Vinner 1994). Another interesting point was that some students wrote the new equation as $ax^2 - bx - c = 0$ because they wrote the original equation as $-ax^2 + bx + c = 0$, in disagreement with the equation written on the working sheet. They thought that if the coefficient of x^2 is negative, it must be written as -a. For them the letter "a" cannot symbolize a negative number. The intuitive way of thinking of these students connects automatically a negative number with the minus sign, which must, in their opinion, be expressed visually. This is one of the aspects of the difficulties students have with formal symbols.

Category III (fourteen students = 11 + 3) Students who did not answer at all or did not know how to write the new equation.

Step 2 Here the students had to conclude from the form of the graph an algebraic inequality.

Almost all students did it correctly $69 = 61 + 8$ and were included in Category I. Four students who checked a specific example and pointed out the correct pair of inequalities which fitted the example were classified under Category II, because they did not handle the general situation, though, these students showed in this example that they knew how to make the right transition from a graphic representation to an algebraic one.

Step 3 The students had to substitute first $x = 1$ and then $x = -1$. Here, sixty-one students $(50 + 11)$ wrote down the right answer and were classified under Category I. Eight students were classified here under Category II as they either omitted the c or wrote; $f(-1) = a - b - c$. This might be clerical errors or that the students thought in the first case that as the summand c is not influenced by the substitution, it can be omitted and in the second case, that the substitution -1 changes the sign of c, too. If so, we see here again a certain difficulty in working with formal algebraic symbols. In category III we included four students $(3 + 1)$ 3 did not answer the question at all and one gave an interesting answer. He wrote $a^2 + b + c > 0$ and $a^2 - b + c > 0$. The square sign in x^2 which is of course crucial here, triggered some mechanism which led the student to sustain it, even after a numerical substitution was carried out.

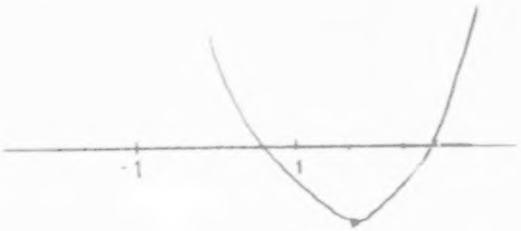
Step 4 In this step, as in step 2, the students had to show an ability to pass from the form of the graph — the vertex of a parabola lies between the roots — to the algebraic inequality which expresses this fact: $-1 < -b/2a < 1$ which is equivalent to $|b| < 2a$ (as $a > 0$). Here, twenty-six students $(22 + 4)$ answered correctly and were included in category I, thirty-eight students $(30 + 8)$ pointed out the correct inequality, but without any reasoning. They were classified under category II. Nine students were classified under Category III as they either did not answer at all or pointed to a wrong inequality. In our opinion, the problem here lies in the difficulty students usually have dealing with absolute values.

Steps 5-7 of the task were given (because of technical reasons) only to 37 students including the 12 prospective teachers.

Step 5: The students had to remember that in case of a single root, this root is the vertex and the relevant inequality is $|b| < 2a$. Here seventeen students $(11 + 6)$ answered in this way and were included in Category I. Other fifteen students $(11 + 4)$ pointed to the correct inequality without giving any explanation and were classified under Category II. Additional five students $(3 + 2)$ gave wrong answers and were classified under Category III. Here we encountered a good example of what we call a "pseudo analytical" argument. One student answered as follows: "Since if a parabola has a single root, then $b^2 - 4ac = 0$; and since in this formula, the sign of b is irrelevant, and since the only inequality where the sign of b is irrelevant is the inequality $|b| < 2a$, hence this is the appropriate inequality".

Step 6: We expected this step to be rather difficult for the students as it involved the concept of the "complement of a set" and required good visual thinking in order to point out all five cases. Very often students confuse the concept of the complement with the opposite attribute. Therefore, we expected many answers to point out only the cases where both roots were outside the interval (-1,1) and omit the cases of one root inside and one root outside. Thirty students (23+7) students drew correctly all five cases and were of course included (to our pleasant surprise) in Category I. Two students pointed only to the above mentioned cases and were included in category II. Five students did not answer at all and were classified under Category III..

Step 7: In this step the students had some difficulties in understanding which inequality must be violated. Only 19 students (16 + 3) students gave a full answer and were included in Category I. Nine students (7 + 2) answers were classified under Category II. These answers were only partially correct. The students pointed only in part of the cases to the crucial inequality. A typical error was the following one: In the case of one root inside the interval and one root outside, the students drew the following graph:



In this drawing, the vertex is outside the interval, but it is not necessarily there. The crucial inequality which is violated is of course $f(1) > 0$ and not $|b| < 2a$ to which the student pointed. Tall (1994, p.38) writes that one aspect of advanced mathematical thinking is to use images "without being enslaved by them". There is nothing negative in using visualization to draw algebraic conclusions, but the student who possesses good mathematical thinking knows what general conclusions he can draw and which parts of the graph hold only in the specific example. Here, students showed that they do not possess this aspect of advanced mathematical thinking according to Tall. Nine students (2+7) did not answer at all and were included in Category III.

Step 8: The role of the last step was to check if the students know how to apply the characterization they had just learnt to a particular case. Here they had to add also the requirement of a positive discriminant. They had to solve a system of inequalities, among them an inequality including absolute values. Half of the students ($36 = 34+2$) did the computation correctly (disregarding some minor errors) and were classified under Category I. Ten students ($6+4$) solved correctly only some of the inequalities and were classified under Category II. They had, of course, difficulties solving the inequality $|k+3| < 10$, and actually solved only the inequality $k+3 < 10$. Three students had difficulties with the inequality $f(-1) > 0$ as the parameter k vanishes and they did not know what to conclude about k from the inequality $0.2 > 0$.

Here we see some of the difficulties students have regarding the role of parameters. Twenty-seven students ($21 + 6$) either solved only the inequality requiring the discriminant to be positive or did not answer at all. We classified all these answers under Category III, since only to check when an equation has two roots is a common exercise and the crucial inequalities which are the outcome of this self-learning task were not even mentioned by the students.

Summary: In our summary we want to do an overall analysis of the students' performance, according to our three categories. Since steps 2,3 and 6 of the task were rather easy (more than 80% were classified under category I), our analysis will be based mainly on the students' performance in the remaining steps. These steps are more interesting from the cognitive point of view. We attach much importance to the last question, as we believe that not knowing how to apply the criterion which is the goal of the entire task is a failure.

Using this kind of analysis we conclude that, on the average, about 35% of the students were included in Category I, about 40% were included in Category II, and the rest were included in Category III. In the last step, 37% of the students were included in Category III. Thus, we can claim that about 60% - 70% of the students in our sample are able to follow a self-learning task. They will not do it in a faultless way but they can grasp the general idea. They can overcome their tendency to avoid new problems and possess an adequate level of "relational understanding."

One of the factors which prevented students in our sample from performing better is their poor handling of formal algebraic expressions and rather complicated algebraic manipulations like solving inequalities containing absolute values.

Students showed in most cases that they can make the correct transition from graphic aspects to algebraic ones. They do not avoid visual consideration as do first-year university students according to Vinner (1989).

The school's students were somewhat better than the prospective teachers, especially where technical computations had to be made. They are probably better trained in this area.

The majority of the school's students enjoyed very much working on the task as it was for them a pleasant deviation from the routine and an interesting challenge. They asked for more tasks of this kind.

Our data and analysis justify suggesting this kind of activity as part of the common mathematical practice for students with similar qualifications to those in our sample. Namely, students with reasonable motivation to perform a multiple step task.

References

Schoenfeld, A.H. (1987): "What's All the Fuss About Metacognition" in: A. Schoenfeld ed., "Cognitive Science and Mathematics Education" Lawrence Erlbaum Pub., p.189-216.

Skemp, R. (1979): "Goals of Learning and Qualities of Understanding," *Mathematics Teaching*, 88.

Tall, D. (1994): "The Psychology of Advanced Mathematical Thinking," handed out at the working group on A.M.T, P.M.E., 18 Lisbon,

Vinner, S. (1989): "The Avoidance of Visual Consideration in Calculus Students," *Focus on Learning Problems in Mathematics*, Vol. 11(2), p.149-156.

Vinner, S. (1994): "Traditional Mathematics Classrooms: Some Seemingly Unavoidable Features," Proceedings of P.M.E. of Lisbon, Vol. IV, p.353-360.

Learning about Teaching: The Potential of Specific Mathematics Teaching Examples, Presented on Interactive Multimedia

Peter Sullivan
Australian Catholic University

Judith Mousley
Deakin University

Abstract: *This report arises from a project which is developing particular exemplars to support learning about teaching mathematics which are delivered by computer, integrating video, graphics and text. A small scale investigation of the use of such a resource with groups of teacher education students indicated that students can move beyond merely describing teaching events to detailed analysis and explanation. It is suggested that such a resource can enhance learning about teaching because links between theory and practice are made explicit, students can move to higher levels of cognitive engagement on the study of teaching, and students can assume direct responsibility for their own learning.*

The use of teaching exemplars

Initial education and induction for professionals like lawyers and doctors is less problematic than for teachers since entrants into those professions have had few experiences of the operation of the profession and so their initiation is likely to be more formative (Lortie, 1975). As pre-service teacher education students begin their training with well developed conceptions of teaching, it is likely that they will interpret theoretical positions and suggested teaching approaches in the light of their own pre-conceptions and prejudices. The tendency will be for reproduction of models of teaching to which the students have already been exposed rather than a genuine consideration of alternate approaches. The problem is compounded for in-service teachers.

Clearly there is a need for such teacher education students to engage in the study of teaching in a way which challenges them to evaluate their beliefs and understandings of teaching processes. There are a number of challenges for teacher educators, two of which are the focus of this report. The first is to support the study of, and reflection on, teaching in a way which stimulates not only description of teaching events, but also interpretation and analysis of actions, contexts and interactions. The second is to structure learning opportunities which facilitate teacher education students' construction of their own knowledge about teaching.

Opportunities for linking theory and practice are central to meeting both of these challenges. The practicum provides rich opportunities for this synthesis, but such interaction is often less than ideal because observation of classrooms is carried out by inexperienced observers in varied and isolated situations. In preliminary studies, we found that field experience did not generally allow teacher education students to discuss, analyse, or even identify quality teaching practices (Mousley & Clements, 1990; Mousley, Sullivan & Clements, 1991).

In the project which forms the basis of this report, group study of particular teaching exemplars or cases was used to prepare students for, and to supplement, their practicum. The term case methods is used to refer to the study, analysis and reflection on particular teaching incidents or examples of classroom teaching (cf Barnett, 1991). Merseth and Lacey (1993) suggest that case methods can develop skills of critical analysis and problem solving, represent the complexity of teaching situations, foster multiple perspectives and levels of analysis, and offer students opportunity to engage directly in their own education.

We used interactive multimedia (IMM) to present the source material for the case studies. Prior to the development of the IMM resource, this project identified, and developed a framework for describing, elements of quality teaching. It used analysis of recent literature and a survey of 200 practitioners, teacher educators and other education professionals from several countries (Sullivan & Mousley, 1994). Examples of full lessons which exemplified the components of quality teaching were partially scripted, then taught and videotaped. These tapes were examined using several techniques, including a qualitative analysis of unstructured reviews of the lessons by over 30 experienced teacher educators (see Mousley, Sullivan, & Gervasoni, 1994), then transferred to CD-ROM disc and an IMM environment was authored to provide flexible access to these data.

The IMM resource is presented on computer, and includes videotapes of mathematics lessons, other video records such as pre and post lesson interviews with the teachers, procedural documents and readings associated with the lessons, graphic representations of data, and other appropriate resources. It was not authored as a didactic "This is the way to teach!" program, but is an extensive information bank which can be accessed in flexible ways to support detailed classroom observation and analysis. Indexing and the interactive nature of the program allows scenarios to be accessed and linked to other data at the press of a key, thus enabling users to focus on specific teaching skills, moments of interaction, selected sets of incidents, sequences of events, links between written theory and action, and so on.

Merseth and Lacey (1993) argue that the potential of multimedia includes the possibility of introducing the complexity of teaching to novices, that the non linear capability distinguish multimedia from conventional videotape since they allow the use of multiple perspectives and opportunities to review situations, and that knowledge arising from the use of multimedia is obviously constructed by the user.

Multimedia programs have been used for didactic delivery of teacher education programs (eg. Carlson & Falk, 1991, Pape & McIntyre, 1992), including some specifically for classroom observation (eg. Dgasy, 1991; Jacobson & Hafnr, 1991). Research summaries (see, for instance Widen & Roth, 1992) report significant changes in the roles of students and teachers in computer-aided learning situations—but the process remains essentially one of following guided instructional pathways. The current project focuses on provision of an information base from which knowledge can be constructed by users.

This is a report of a small scale trialing as a preliminary investigation of the effectiveness of the resource.

A study of the effectiveness of the study of teaching cases delivered on interactive multimedia

This study had two goals. The first was to describe some teacher education students' interactions with the resource, and, in particular, whether users described teaching events observed or whether they endeavoured to interpret and/or analyse the events and their significance. The second goal was to examine whether unstructured and unsupervised use of the resource facilitated students' engagement in exploring and discussing teaching events in a way which enhanced the formulation of their own theories about teaching, and which had the potential to challenge their own pre-conceptions.

The subjects in this study used the resource as a work requirement of a final year undergraduate course on the study of mathematics teaching. All interaction with the IMM resource was in groups. After using the resource for approximately 10 hours, the students participated in debriefing seminars and individual interviews. The course also included more conventional components such as reviews of the literature and lectures and tutorials on aspects of mathematics teaching.

There were three aspects to the data collection. First, an observer made unstructured notes during each small group session on both the interaction of users with the hardware, and also on the style and quality of their discussions and critique of the issues which arose. Second, users maintained a journal, compiled individually, during their interactions with the resource. This was essentially an unstructured record of the issues which arose which were seen as significant by the users. These data were supplemented by interview transcripts. Third, a computer log was made of pathways used by the groups through the resource.

Results

The data are predominantly qualitative and only brief excerpts are presented here. The following extract, which is representational of the type of data collected, describes a particular incident and subsequent events from the interaction of one of the groups with the IMM resource, and is illustrative of the style of interactions which occurred. These data suggest that the study of particular cases using IMM does have the potential to engage students in explaining and analysing teaching events and constructing their own knowledge about teaching.

A particular incident

This incident described here arose from one groups' discussion arising from a screen which consists of a map of the classroom with individual tables labelled with the names of the pupils. By placing the mouse over the name of a student it is possible to see a photograph of the pupil. By clicking on the photograph, video clips of each interaction in which that pupil participates can be viewed.

The following is the extract from the record of the observer:

> There was one incident where the group studied a single screen for 18 minutes. During discussion they raised the issue that the boys were better placed. They noted that all five boys out of 16 students were seated around the outside of the room. It was hypothesised that this gave them a better view of what is happening in the classroom.
>
> The group considered other aspects of the grouping of children which arose from the study of the classroom layout. This included such matters as whether boys should be placed together or separately. They pursued this line and investigated the quality of responses that the boys made for particular tasks. This group continually referred to classroom layout in subsequent investigations and used the pause and replay facility again and again during various video excerpts to comment on aspect of classroom layout and design.

The following are extracts from the journal of one student from the above group describing the study of this screen:

> Went to the "Map of Classroom layout". Discovered seating of tables–boys and girls, 5 boys and 11 girls, 4 tables all have four children, some have 2 boys/2 girls, one has no boys.
>
> We got the impression (the teacher) asked lots of questions to boys. Boys appear to be better positioned on outside of tables—better able to see board.
>
> Looked at photos of individual children to see if they were advantaged/disadvantaged by position in room. Decided boys have better view of blackboard/front of room. Feel four groups of four is good but wonder why boys are placed where they are. How are groups selected?
>
> Tables good set up to talk. Tables angled and board not really used so perhaps set up not bad.

In this example, the group focussed on classroom configuration, and used this as the basis of further investigation of associated issues of classroom interactions. Some issues identified were perhaps a reflection of their other studies (eg the gender issue), but other aspects (eg the view of the blackboard) were most likely initiated by the group members themselves.

As a group, the users decided to develop their knowledge of classroom interactions and questionning by focussing on all of the questions asked during these particular lessons. This was recorded in the journal of the same student as follows:

> Would like to look at questions used so went to "Phases of Lesson", then "Extending The Problem". ...
>
> We continually paused program to discuss and record.
>
> Decided to move out of this phase of lesson because we were not clearly focussed on original; questions about her questioning. We were looking at too many different things and getting distracted.
>
> Like to look more at teacher communicating, questioning etc. Want to look at gender breakdown of interactions. Went to "Six components of quality teaching". Chose "communicating"–computer responded with some examples. "Communicating" menu–decided to look at the "Teacher to student communication"–still want to concentrate on boy/girl breakup. Still not sure if this section is what we need.
>
> First 3 questions to girls. Are we getting all communication or just a selection?
>
> Back to main menu–decided to look at "Movies of segments of lesson" and then do manual count.
>
> Looked at Area lesson and Investigation segment. Moved forward with FF to find spot where she was questioning kids. Very handy.

Clearly the group was using the resource to collect data for their investigation. As an aside, it is interesting how the research tool, in this case the IMM resource, can impede thinking about the substantive research issue.

The following are further extracts from this journal which show how this investigation developed. This student went beyond merely describing what was happening and sought to explain and even analyse the events. It is also clear that this student is engaged in a meaningful process of constructing knowledge about teaching:

> Why did she ignore all the hands when setting the kids to work? Did they know what to do? Were they seeking help?–Did ask one girl when put others off.
>
> Interesting how she got the kids quiet by just standing at front of room. Chose a boy to give an explanation of an answer/strategy–>positive response. Questions to girls are more basic eg: Do you understand? Perhaps we should be looking at higher/lower order questioning. Is it reinforcing boys are better at working out? Although we have a feeling (the teacher) wouldn't. Second child chosen to give strategy/answer was also a boy positive response. Third person was a girl. Count came up 10 girl to boy 5, considering ratio of girls to boys in the class this is a fair breakdown.

The group went beyond merely counting interactions and was investigating whether the quality of interactions are related to gender. The group then focussed on graphs of the number of interactions of each pupil and the times of those interaction which are included in the resource

> Decided to look at matching phase in other lessons. But we can't find one to match. Went back to check last menu. Decided to look at Reporting Back segment of lesson.

> Comparing strategies used to answer questions. Chose Emily first, then Olivia, then Laura. "Data from lesson" gave graph of interaction. (The resource contains graphs which show the number of interactions of each pupil, as well as the times of each interaction.)
>
> Trying to find graphs again. ... (A group member) asked "Do we know where they are sitting?" "Map of classroom layout" shows John up near front, Daniel in back corner. Both very close to camera. Why didn't they record any interactions? Clicking mouse button on layout to give picture of pupil is handy to remind us of their part in lesson.
>
> Wondered if positioning in relation to camera had influenced the way children participated, or seemed to participate. Go back to graph to see who participated the most, Patrick, Emily, Katlin, Gabrielle. Gabrielle and Katlin on front table, Patrick and Emily on back table. Positioning doesn't seem to influence number of interactions.
>
> Decided to look at each student's interactions. "Movies of each student's interactions" in research menu. Click on child's photo to get video of interactions. Instructions on screen helpful. Quality of video is very good. "Video of children's interactions" watched Patrick, then thought it would be interesting to watch Emily. Emily's questions seemed designed to get attention–eg: asking questions about rules that had been clearly explained.

This student, at least, was engaging in a genuine research process and was seeking to explain and analyse teaching processes. In addition to comments on program design, this user addressed a range of issues associated with classroom interactions, including classroom seating and its link to classroom questioning, group structure and composition, different types of questions asked to boys and girls, the use of target students to facilitate lesson flow, differential class control mechanisms based on gender, the use of high and low order questions, the relationship of seat placement to style of interaction, and identification of attention seeking behaviour. While the user did not necessarily move towards closure on any issue, the consideration of classroom variables is probably sufficient indication of the potential of this resource to support learning about teaching. It also appears that the interaction with such case material in small groups engages teacher education students in thinking about teaching in a way which has the potential to challenge their own conceptions and which can facilitate their own construction of teaching processes.

Other issues

Other issues related to the use of IMM and case methods were identified in this trialing. Some of these are referred to briefly in the following.

While the resource allowed high level of cognitive engagement for some, there were other students whose reports were restricted to merely describing events presented, such as:

> Confusion over solid shape and dimensions therefore discusses notion of dimensions. Perhaps a cube could have been drawn. Recaps directions. Strategy for drawing design was open.

> Discussion confusion of box shape. Strategies for drawing–very open–not too much direction. The diagram not very descriptive for 3 dimensions. Expectations and explanations of team work not given.

Teacher educators need to find ways to support such students in moving beyond mere description, and attempting to seek to explain and/or analyse classroom events.

It also appears that there may be a need for some intervention to allow students to articulate concerns and also to consider alternate perspectives, as perhaps indicated by this comment from one journal:

> One point I would like to make at this stage of viewing is that I am wondering what exactly is being taught. I am a little concerned nowadays that pupils are not being taught much ie teachers not teaching anything, but rather children finding out for themselves idea.

While this indicates a prior conception of teaching processes, it is also a significant issue for many teachers. One of the advantages of the use of case methods is that such fundamental underlying conceptions can be raised by students, and can then be addressed by the teacher educator at a time appropriate for the students.

It was also clear that the use of groups is far from problematic and strategies for supporting group work need to be identified, as indicated in this journal entry:

> I mentioned I think in the last diary entry that I felt frustrated. Well I feel even more frustrated now with these two women not being able to make their minds up on what to look at let alone talk about! Maybe they feel the way I do! I think it's better to view the entire lesson then talk about the elements of quality teaching rather than view little bits at a time. It is far too disjointed that way. Perhaps we need more structure.

It also seems that while some students thrive on the freedom provided by such a resource, other prefer more structure to their work:

> The program has some fantastic features but we tended to lose track of what we were doing and constantly went back and forth between menus. More useful if we had a more defined research question and were experienced with the program.

In summary, this small scale investigation of the use on an IMM resource suggests that case methods have potential to support the integration of theory and practice, and that IMM can be an effective way to present case materials to teacher education students. While some students were able to engage in the analysis of teaching in a significant way, other students required more support. It is clear that there is a range of issues related to the levels of student support and direction, and the form of any intervention, which require further research.

References

Barnett, C. (1991). *Case Methods: A promising vehicle for expanding the pedagogical knowledge base in mathematics*. Paper presented at the 1991 Annual meeting of the American Educational Research Association, Chicago, April 3-7.

Carlson, H. L., & Falk, D. R. (1991). Effectiveness of interactive videodisc instructional programs in elementary teacher education. *Journal of Educational Technology Systems, 19* (2), 151-63.

Dgasy, G. (1991). A comparison of methods for providing evaluative feedback to practicum students on segments of teaching. *Proceedings of the Rural Education Symposium* (pp. 77-89) Nashville: Association for Rural Education.

Jacobson, L., & Hafner, L. R. (June, 1991). *Using interactive videodisc technology to enhance assessor training*. Paper presented to the annual conference of the International Personnel Management Association Assessment Council, Chicago.

Lortie, D. C. (1975). *School teacher: A sociological study*. Chicago: University of Chicago Press.

Merseth, K. K., & Lacey, C. A. (1993). Weaving stronger fabric: The pedagogical promise of hypermedia and case methods in teacher education. *Teacher and Teacher Education*, 9(3), 283-299

Mousley, J., & Clements, M. A. (1990). The culture of mathematics classrooms. In M. A. Clements (Ed.) *Whither mathematics?* (pp. 397-406) Melbourne: Mathematical Association of Victoria.

Mousley, J., Sullivan, P. & Clements, M. A. (August, 1991). Student teachers' perceptions of the way mathematics is taught in their teaching practice schools. In *Proceedings of the 1991 conference of the Mathematics Education Lecturers' Association*. Perth: MELA.

Mousley, J., Sullivan, P., & Gervasoni, A. (1994). The analysis of teaching: Constraints in lesson description and critique. In G. Bell, B. Wright, N. Leeson, & J. Geake (Eds.),*Challenges in mathematics education: Constraints on construction* (pp. 437-444). Lismore: Mathematics Education Research Group of Australia.

Pape, S., & McIntyre, D. J. (February, 1992). *Utilising video protocols to enhance teacher reflective thinking*. Paper presented at the annual meeting of The Association of Teacher Educators, Orlando.

Sullivan, P., & Mousley, J. (1994). Quality mathematics teaching: Describing some key components. *Mathematics Education Research Journal, 6* (1), 4-22.

Widen W. C., & Roth, G. L. (1992). Instructional technologies in the workforce. *Journal of Industrial Teacher Education, 29* (2) 65-85.

MATHEMATICAL MODELLING IN THE SCIENCES THROUGH THE EYES OF MARINA AND ADAM

Rosamund Sutherland[1], Teresa Rojano[2]
Simon Mochon[2], Emanuel Jinich[2] and Susan Molyneux[1]
[1]School of Education, University of Bristol, UK
[2]Centre for Research and Advanced Studies, IPN, México

This paper presents the results of a Mexican/British research project which investigated the mediating role of spreadsheets for expressing and solving mathematical modelling problems within science. The study was carried out with two groups of 16-18 year old students, one in Mexico and one in the UK. Results of the study indicate that the external represesentations of the spreadsheet offer students a structuring resource for developing mathematical models in science and for making sense of the links between the model and the physical situation. Students' preferences for algebraic and graphical representations were shown to be strongly influenced by previous school experiences.

Introduction and Theoretical Approach

Research on mathematical modelling in education has recently received more attention due to the widespread use of computers (Mellar et al, 1994). Within this paper, we report on a collaborative Mexican/British study aimed at investigating the mediating role of spreadsheets for expressing and solving mathematical modelling problems within science. We also investigated the links students make between a physical situation and a mathematical model of that situation, and probed their evolving preferences for the external representations available in the spreadsheet. The study was carried out with a Mexican and a UK groups of 16-18 year old students who were all studying one or more of the sciences prior to taking University entrance examinations.

There are a number of interrelated aspects of Vygotsky's work which influenced the research. One of them is the idea that it is "the person-acting with mediational means" which is the focus of the analysis (Vygotsky, 1978; Wertsch, 1991). From the point of view of this study, the spreadsheet environment and more specifically the spreadsheet external representations are potential mediators of mathematical modelling processes. Vygotsky stressed that mediational means are sociocultural in the sense that mediated action cannot be separated from the social milieu in which it is carried out. This implies that studies of school pupils cannot be separated from influences such as school curriculum, school culture and out of school culture.

The work of Lave (1988) suggests that mathematics can surface in different forms within different settings, and that mathematics can give structure to, or be structured by, other ongoing activities. With this idea in mind, we wanted to probe how mathematical practices structured science courses in two school cultures, which structuring resources were manifested in student modelling activities and whether a spreadsheet could be used as a structuring resource by students.

Methodology

The study consisted of three phases during which, case studies of 9 Mexican and 12 UK students were developed (although all experimental work involved whole classes rather than just case-study students). Phase 1 involved a study of mathematical practices within science before the spreadsheet was made available, followed by a pre-evaluation test and follow-up interviews. In the first part of the school year, a training course in Excel was given to the students, using simple, yet relevant science problems. Phase 2 included the implementation and observation of experimental spreadsheet modelling activities (developed by the teams) which students worked on during appropriate science lessons, in the classroom. Further interviews were conducted. The final phase entailed a post-evaluation test and interviews. Data collected included: field notes and video recordings of mathematical practices before the spreadsheet was made available; field notes and video recordings of innovatory modelling sessions; video recordings of individual interviews with case study students; and records of students' paper-based and computer-based work.

To match the topics of the science curricula in each country, the two groups worked on slightly different sets of modelling problems. The titles of the worksheets designed for the modelling activities were : "Diffusion", "Population Growth" and "Population Genetics" for the biology class; "Chemical Equilibrium", Environmental Pollution", "Periodicity" and "Lattice Energies" for the chemistry class and "Collisions I (Inelastic)", "Collisions II (Elastic)", "Gravitation", "Artificial Satellites" and "Oscillations and Waves" for the physics class (the underlined ones were common for both countries). The purpose of these activities was to allow the students to create an "artificial world" (Mellar et al, 1994) as an image of phenomena, so they could be explored and studied in detail. These models were a combination of "exploratory" models (models which involve the learner in exploring ideas about a topic presented by someone else) and "expressive" models (ones which involve learners in expressing their own ideas) since the students built the models guided by the presentation of the worksheets (Mellar et al, 1994). The students were asked to construct graphs relating the variables of the model and to play with various parameters corresponding to different physical situations.

Within this paper, we present some of the results of the study, illustrated in places by the views and work of two of the students that participated in the experience: Marina, a Mexican student, and Adam, an English student.

Mathematical Practices before spreadsheet work

From our classroom observation at the beginning of the school year, before the computer work commenced, we found differences between the teaching and learning of science of the two groups of students. The Mexican students were taught in a formal way, referred by some authors (diSessa, 1993) as a top to bottom approach: from general ideas down to particular examples. In contrast, the UK students were exposed first to practical situations and specific examples with the objective of investigating and discussing the important issues of the topic being studied, leading to a general view. Analysis of the pre-16 curricula and examinations in both countries also indicates quite substantial differences in the types of teaching methods emphasised and the approaches to mathematics. More algorithmic methods are emphasised in mathematics in Mexico and in particular formal algebraic approaches. However, in the UK there has been a move away from taught algorithms resulting in an emphasis on approximate answers and the reading of graphs.

Analysis of the pre-evaluation and interviews showed that the Mexican students demonstrated a preference for, and proficient use of, formulae and equations whereas the UK students preferred, and were more competent with graphs. Mexican students tended to use taught formal algorithmic methods when solving science problems, for example 'the rule of three'. They used these external resources even when interpreting tables and graphs. The result in Mexico is typified by Marina who described a preference for symbolic representations rather than tables or graphs and frequently used the 'rule of three', even in cases where it was not appropriate. For example, in the pre-evaluation when requested to read-off values from a non-linear graph. The use of such a taught algorithm seems also to be related to a need to provide an exact answer. In contrast, Adam demonstrated a strong preference for graphical representations, both for solving problems and as a means of understanding physical phenomena. This preference stemmed from his belief that graphs allow a global view ("it is more visual and you can see all of it"). An equation seems not to be used by Adam to give insight into understanding a physical situation whereas he would use a graph because, "it's easier to see what's happening".

Whilst, like Marina, Adam was competent with algebraic manipulations, he did not demonstrate the same need for exact answers or the use of algorithms. Adam was happy to approximate answers when he thought this was appropriate and used a graphical representation as a tool for this purpose. He articulated a dislike of algorithmic approaches to problem solving as "you don't understand why you do each thing". In particular, he said that he did not like using a "triangle method" for manipulating three element expressions (e.g. F = ma) which they had been taught in science as a rote method for transforming equations.

Neither of the groups had experienced mathematical modelling in science before the study, although a sub-set of the UK students (who were also studying post-16 mathematics) were being introduced to modelling in mathematics. In both Mexico and the UK, before the spreadsheet activity, most students had difficulty articulating their understanding of a mathematical model and several professed to have never heard of the idea. In his first interview, Adam said, "is a model something to explain how something works, in a more clear way ?".

Spreadsheet modelling

The vast majority of students in both Mexico and UK became competent at developing spreadsheet models of science situations and developed some understanding of mathematical modelling. When the Mexican students were asked in the final interview what they understood by a mathematical model, they gave a partial list of the names of the spreadsheet activities carried out during the year and there was evidence that the spreadsheet experience had influenced their conceptions. For example, one pupil stated that it is "a description of a real situation using numbers and variables" (a spreadsheet type of idea). The UK students made comments such as, "it's a theory ... the model isn't actually true"; "it's mathematical expressions which describe the situation - what is happening".

To a certain extent, the student's initial preferences for spreadsheet representations reflected their preferences in paper-based situations. For example, Adam's preference and use of the graphical representation. His preference did not diminish as the year progressed but the process of actually constructing a graph within a spreadsheet may have played a significant role in his approach to understanding a topic. For example, he said that he disliked one model because "I found that a bit boring, I knew what to expect 'cos we'd seen all the graphs in the book". However, his spreadsheet work meant that, "I'd remember it (the graphs) more vividly than I would if I saw it in a book". Marina's appreciation of the graphical representation developed through the school year so that she no longer relied solely on symbolic representations but was able to incorporate other representations when making sense of a science problem. Research in psychology has suggested that students have a working style along the dimension visual/sentential (Cox & Brna, 1995) Our study using a modelling approach with spreadsheets suggests that such a style would be very influenced by previous school experience.

The spreadsheet approach facilitates the analysis of a science problem. The different representations available (tables, graphs and formulae) allow students to produce a rich spectrum of results and lead students to make links between them. The transformation of formulae into lists of numbers and graphs provokes students to face a variety of representations of a phenomenon, in a point by point or global form. The spreadsheet allows a student to change the values of the parameters involved and to see immediately the effect on the tables and graphs, giving a student the power to analyse many cases with a simple change of a number in a cell. The spreadsheet work confronted students with the need to make links between representations and they began to understand the relationship between the table and the graph and the spreadsheet formulae. They appreciated that altering one aspect of a formula had implications for the other representations so for example, with reference to the parameter 'a' in the formula $x = at^2 - t^3$ Adam said "Define the constant, so you can change it and the whole thing (spreadsheet) will change".

The spreadsheet as a structuring resource

For the students, the spreadsheet became an important tool not only to perform calculations, but also to give structure to the model and to provide feedback on the modelling process. In the construction of a spreadsheet model, the variables are usually defined in columns and their relationships with each other have to be made explicitly with spreadsheet formulae. Analysis of videos of students constructing spreadsheet models indicates that they can use a spreadsheet as a structuring resource when identifying the variables within a model, "This column is going to be (pointing to the column) the amount of substance.....does it understand formulae....at a time...so N is the amount we came out at the end ... so it's that which is N_0 which is the 100 ". Using 'Define Name' to separate absolute from relative spreadsheet references helped students to differentiate between the parameters and variables within a formal model.

The ways in which the spreadsheet structured Marina's modelling activity is illustrated by her solution to a problem presented to the students in the final interview,

> *The position of a car as a function of time is given by :* $x = at^2 - t^3$ ...

followed by questions such as,

> *If a=9, what is the car's position at t=8 sec.?... describe what happens between t=6 sec. and t=9 sec. ... what happens at t=6 sec.?... what happens when time increases? Analyse the situation when a=12...*

Marina worked on this problem on a spreadsheet although, on occasions, she complemented this tool with a calculator to perform some operations. She demonstrated the ability to pull information from all the resources available to her: the given equation; the table of values; the graph constructed by her in the spreadsheet. Although initially she started to use an inappropriate formal rule ($v = d/t$) to give the velocity of the car at $t=6$ (where the graph of position shows a maximum), after an intervention from the interviewer she realised that there was a maximum at this point in the graph and therefore she concluded that the velocity must be zero, applying a physical argument: "it is zero, because it (the car) goes backwards". This suggests that formal mathematics is a strong structuring resource for Marina when scientific problem-solving is the ongoing activity. Adam also demonstrated adaptability in the use of mathematical resources to solve this problem. He moved between representations, for example using the spreadsheet table to read off one answer and then estimating the answer to a different question from the graph. He also switched between resources, e.g. taking approximate readings from the graph then checking this estimate with arithmetic on a calculator. Because understanding the visual is an important aspect of science for Adam, he immediately correctly interpreted the meaning of a maximum in the speed-time graph and did not attempt to use a more algebraic approach. For Adam, the science (or the graphical representation) is the dominant structuring resource.

By the final interview Adam had incorporated the spreadsheet as a problem-solving resource, choosing a spreadsheet to solve a problem where "I was just going to do it manually, myself, but I decided it would be easier to use a spreadsheet as well". In this situation it seems that Adam used the spreadsheet to support him in the construction of a general formula, which he had found difficult to do from his specific calculations on paper.

The spreadsheet as a mediator between the mathematical model and the physical situation

Mathematical modelling requires movement between the physical situation being modelled and the mathematical representations of that model. The format of the spreadsheet keeps alive the scientific content with the labelling of columns and graphs, building on the conceptual development of the students. The pure mathematical (algebraic) approach on the contrary, separates the context and puts a heavy burden on the mathematical manipulations needed to solve a problem. We conjecture that the dynamic nature of the spreadsheet feedback provoked more of a move between the physical situation and the formal model, than similar work with paper. If students could not make sense of the feedback from the spreadsheet table and / or graph with respect to their formal or everyday knowledge of the physical world, they then questioned the formal spreadsheet model which they themselves had constructed. The fact that they had entered the spreadsheet formulae themselves may be critical here. In other words constructing a spreadsheet model provokes movement between, and a re-questioning of, the formal-physical relationship.

Some concluding remarks

The spreadsheet seems to offer students a new psychological tool (Wertsch, 1991) for developing mathematical models in science. The computer feedback from the formal spreadsheet model provoked students to re-examine their assumptions about the physical situation.

Our work suggests the possibility of influencing mathematical practices within science subjects, using a modelling approach embedded into a computational environment such as spreadsheets. The spreadsheet approach helped the Mexican students to appreciate the graphical and numerical representations and at the same time helped the UK students to make sense of the algebraic representation of the models.

diSessa suggests that the top-down approach to science teaching only works if students already have a well-developed set of relevant primitive experiences from which to draw upon (diSessa, 1993). The bottom-up approach can result in some students finding it difficult to make sense of formal algebraic models in science. White (1993) puts forward an argument for a middle-out approach in which students are introduced to scientific ideas through causal models introduced at an intermediate level of abstraction. We suggest that spreadsheet models present an intermediate level of abstraction which enable students to move between the formal model and the physical situation.

References

Cox, R. & Brna P (1995) Supporting the Use of External representations in Problem Solving: The Need for Flexible Learning Environments, Researvch Paper 686, Dept of Artificial Intelligence, University of Edinburgh.

diSessa, A. (1993) Towards an Epistemology of Physics; *Cognition and Instruction*, 10, pp. 105-225.

Lave, U. (1988) *Cognition in Practice: Mind, mathematics and culture in everyday life*. Cambridge University Press, Cambridge, England.

Mellar, H., Bliss, Boohan, Ogborn and Tompsett (Eds., 1994), *Learning with Artificial Worlds*, The Falmer Press, London and Washington D.C.

White, B. (1993) Intermediate Causal Models: A Missing Link for Successful Science Education, in Glaser, R. (ed.); *Advances in Instructional Psychology*, 4, pp. 177-251.

Vygotsky, L.S. (1978), Mind in Society: The Development of Higher psychological Processes, Harvard University Press, Cambridge, Mass.

Wertsch J (1991) *Voices of the Mind: A Sociocultural Approach to Mediated Action*, Harvester, London.

We would like to acknowledge the Spencer Foundation for the support given during the project: The Role of Spreadsheets within School-based Mathematical Practices (Grant No. B-1493). Also thanks to staff and students of Escuela de la Cuidad, México and Hampstead School, UK.

PRE-SERVICE TEACHERS' PROBLEM SOLVING STRATEGIES

Margaret Taplin, University of Hong Kong

This paper explores the approaches to problem solving used by a group of twenty pre-service primary teachers. Results indicated that students preferred to work with a narrow range of strategies, predominantly verbal and numerical. There was no particular approach which contributed more than others to successful outcomes.

The shift towards teaching mathematics through problem solving has arisen from changes in many technological and social factors (Resnick, 1987, NCTM, 1989) and has been well-documented over the past decade (NCTM, 1989). Resnick (1987), for example, believes that "school should focus its efforts on preparing people to be good adaptive learners, so that they can perform effectively when situations are unpredictable and task demands change" (p.18). As problem-solving approaches to teaching mathematics are becoming increasingly important, it is becoming clear that successful implementation is largely dependent upon the ability of teachers to incorporate such approaches into their programmes. Lester (1994) highlighted the need to understand more about the teacher's role in teaching through problem solving if this type of teaching is to be successful.

To teach mathematics effectively, it is necessary for teachers to have competence in a complex interaction of knowledge domains: knowledge of and about mathematics itself, about the pedagogy of mathematics, and about the students (Borko, Eisenhart, Brown, Underhill, Jones and Agard, 1992, Eisenhart, Borko, Underhill, Brown, Jones and Agard, 1993, Schifter and Fosnot, 1993, Cooney, 1994, Fernandes and Vale, 1994, Jones, 1995, Swafford, 1995). Applying the above to mathematical problem solving, it can be seen that to teach successfully through this medium, teachers need competence in the same three essential components: knowledge and understanding of what a problem solving approach to teaching mathematics is (content knowledge), knowledge of and success in using strategies for solving problems for themselves (procedural knowledge), knowledge about how children can learn mathematical concepts through problem solving (content-pedagogical knowledge) as well as a fourth component, belief in the value of using problem-solving approaches to teaching mathematics. The focus of the project reported in this paper was one aspect of this model, teachers' procedural knowledge of problem-solving strategies.

Knowledge of mathematical problem-solving strategies is, in itself, a combination of several factors: the problem solver's mathematical knowledge,

knowledge of heuristics, affective factors which influence the way the individual views problem solving and knowledge of the managerial skills associated with selecting and implementing appropriate strategies (Schoenfeld, 1985). In addition to knowing the "discrete skills and procedures"(Lester, 1985, p.43) of problem solving, problem solvers need to know how to make "[managerial] decisions about whether to persevere along a possible solution path" (McLeod, 1988, p.138) and they need to be flexible about changing strategies when the time is appropriate (Mason, Burton and Stacey, 1987, Taplin, 1994, 1995). If teachers are to guide their pupils to use these skills, it is first necessary for them to be able to do so themselves, yet little is known about the extent and quality of primary school teachers' knowledge in this area.

In addition to learning more about teachers' knowledge of problem-solving skills, it is important to know whether they have preferences for solving problems in particular ways. There is evidence that problem solvers do not necessarily all use the same methods for processing information (Krutetskii, 1976, Shama and Dreyfus, 1994). The methods they use can include verbal-logical and visual-pictorial (Krutetskii, 1976), physical/kinaesthetic, ikonic or notational forms, or various combinations of these (Gardner, 1983, Thomas and Mulligan, 1994). While Mayer & Sims (1994) found that some students do not need visual prompts because they can generate their own representations, others have reported the manipulation of materials (Owens, 1994), drawing diagrams (Resnick and Ford, 1981), or a combination of these (Bishop, 1983) to be important in successful problem solving. Krutetskii (1976) claimed that students can be equally successful at mathematics with different correlations between visual-pictorial and verbal-logical components. Watson, Collis and Campbell (1994) commented on the need for all of these forms to be used to support instruction in schools. Jacobsen, Eggen, Kauchak and Dulaney (1985) reported on research findings that one of the characteristics of superior teachers was that they were able to use a variety of teaching methods to "increase the number of matches between teaching and learning styles" (p. 176). Thus, more needs to be known about teachers' preferred methods for problem solving, to gain some insights into whether they are likely to be compatible with the learning styles of the children they teach.

In an endeavour to learn more about teachers' preferred methods of processing problem-solving information, and their management strategies, this study focused on a group of pre-service teachers. With a view to addressing any emerging gaps in their skills, the following questions were investigated.

What are the pre-service teachers' most commonly preferred methods for processing problem-solving information?

Is the use of any particular method more likely to contribute to successful outcomes than other methods?

Are there patterns in the way the pre-service teachers' manage their use of strategies, and do these contribute differently to successful outcomes?

Methodology

The data used to explore the above questions were collected as part of a long-term project to develop and evaluate a computer-assisted tutorial (James and Taplin, 1994, Taplin and James, 1994). The aim of the tutorial was to address deficiencies in the mathematical knowledge and problem-solving skills of pre-service teachers, as a supplement to their mathematics education classes. The tutorial presented basic mathematical knowledge embedded in problem contexts. It provided series of hints to solve the problems, based on heuristics described by writers such as Polya (1957). The students were able to select from four categories of hints: verbal, which gave extended explanations of the hints in sentence form, numerical, which presented the hints in equation form, visual, which used two-dimensional diagrams, and concrete, which showed three-dimensional representations of the hints. It was possible to collect data concerning the students' preferred methods of processing information, their levels of success, and their strategy management techniques because the tutorial incorporated a built-in tracking system which recorded each hint they chose and every attempted answer.

Sample

The pre-service teachers described in this paper were members of a first year class who had performed poorly (mean score of 4) on a test of twelve problem tasks which embedded knowledge that they could be expected to teach in primary schools. A group of 40 students volunteered to trial the use of the tutorial, which covered six of the more difficult of these problems (Appendix). They worked on the tutorial over a period of four weeks, for as many sessions and as long as they chose to, before completing a post-test of parallel problems which were unfamiliar to them. A random sample of 20 student protocols was selected for detailed analysis of their problem-solving strategies. Of these twenty students, four spent two sessions working on the tutorial, five spent three sessions, and ten did four sessions. One student completed only a single session.

Results

For the total group of students who completed the tutorial, there was a significant difference in the mean pre-test and post-test scores ($t=14.17$, $p<0.001$). There was a similar significant difference for the sample of twenty students described in this paper ($t=9.775$, $p<0.001$), with a range in post-test increments from -1 to 5.

Preferred Hint Categories

To identify the most frequently used hints, a count was made of the total number of hints in each of the four categories used by the 20 students. The data in Table 1 show the proportions of total hints used by all students, broken down by the category from which they were selected.

Table 1: Percentage of total hints used, by question and category

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Mean
Verbal	25.0	44.0	35.8	40.6	30.2	34.4	35.0
Numerical	53.0	24.0	37.7	43.9	30.2	23.7	35.4
Visual	15.6	14.8	13.2	15.4	10.5	16.1	14.3
Concrete	6.3	16.7	13.2	n/a	29.1	25.8	18.22

The results in Table 1 suggest that the majority of the hints which were selected were verbal or numerical. Visual and concrete hints were used only about half as much as the other two categories. Verbal hints were used the most on Problems 2 and 6, numerical hints were used the most on Problems 1, 3 and 4, and the two were used equally on Problem 5.

To investigate the first research question further, the students' protocols were analysed to find whether they consistently used particular categories of hints. In all cases, the students were consistent, across all of their sessions, in using predominantly the same category or combination of categories (Table 2)

Table 2: Number of students by predominant hint categories

Hints	single category	combination of two categories	combination of three categories
Number of Ss	8	8	4
Mean Number of Tutorial Problems Correct	4.1	4.8	5.0
Mean Improvement Between Pre-Test and Post-Test	3.6	4.6	2.0

Eight students chose a single category and used it consistently throughout all of their sessions. Of these, four used verbal, three used numerical and one used visual. A further eight used a combination of two hint categories: concrete-visual (four students), verbal-numerical (three students) and verbal-visual (one student). Four students used a combination of three, with two using predominantly numerical-verbal-visual and two using mostly numerical-verbal-concrete. The data in Table 2 suggest that all of the above approaches led to similar levels of success with the tutorial problems. A series of Mann-Whitney U Tests indicated that there were no significant differences in post-test increments, although the students who used a single or combination of two categories may have shown a slightly greater improvement on the post-test than the students who combined three categories.

Management of Hints

The second part of the analysis was to explore patterns in the way the students approached the problems or managed their use of the strategy hints, and whether any particular patterns were more likely than others to lead to success. The data in Table 3 indicate that there were three different ways in which the students approached the problem set. The first of these, which has been called the *all-problems approach*, was one where the students tried all of the problems in each of their sessions. Four students used this approach. Three of them tried each problem only once per session, while the fourth had several short attempts (4-7 hints per attempt) at each problem, including nine attempts on the Problems 1 and 2, five on Problem 4, and four on Problem 5. The students who used this approach had achieved a mean of four correct problems out of six by the time they had completed the tutorial.

The second management strategy has been called the *selective approach*, because the students only attempted selected problems. Again, they mostly worked continuously on each problem until they got the correct answer, before moving to the next one, and attempted each problem only once or twice each session. Six students used this approach. The mean number of correct solutions obtained by students using this approach was 5.

Ten students used a combination of the two approaches. In the first session, they attempted all of the problems relatively quickly, with a mean of 10.6 hints per attempt. In subsequent sessions, they selected a limited number of problems and spent longer on each of these, with a mean of 17.7 hints. Again, they did not display a tendency to return to the same problem in a session once they had abandoned it. Only one student returned to a problem she had abandoned, and then only once. The students who used this approach had a mean number of 5 correct solutions during the tutorial. Mann-Whitney U Tests suggest that there were no significant differences in mean improvement on post-test scores for students who used these three approaches.

Table 3: Students' methods of approaching problems

Approach	Number of students using approach	Mean number of hints per attempt	Mean number of correct solutions to tutorial problems	Mean improvement in post-test score
All-problems	4	6.8	4	3.1
Selective	6	20.0	5	4.2
All-problems/selective	10	10.6/17.7	5	4.0

It was noted that some students showed a tendency to change from one set of hints to another during their attempts to solve a problem, whereas others would select one set and use it exclusively, repeating the hints several times if necessary. Table 4 shows the numbers of students who used each of these management styles. Two students were excluded from this table because they used the two approaches equally, and were equally successful using both. From the data presented in Table 4, it can be seen that ten of the students were consistently able to get correct answers to the problems by repeatedly using the same set of hints. Five consistently achieved success by changing from one category to another, and sometimes back again.

Table 4: Management of strategies by outcome

Changed Hint Sets		Used Same Set of Hints	
Consistently Successful	Consistently Unsuccessful	Consistently Successful	Consistently Unsuccessful
5	2	10*	1

*3 of these students were successful using this approach on only 5 of the 6 problems

Summary and Discussion

This study explored the problem-solving approaches of a group of pre-service teachers, on the assumption that if we are to help teachers to develop their potential to present mathematics effectively through problem solving we need to learn more about their own skills. The data from this small-scale study indicated that, although the student teachers performed significantly better on a post-test of parallel problems than they did on a pre-test of the tutorial problems, they tended to select a method of approach and not change from that through the tutorial. Numerical and verbal strategies were selected more frequently than visual or working primarily with these. Half of the student group attained successful answers by repeatedly using the same set of strategies, rather than changing from one set to another. The results confirmed Krutetskii's (1976) claim that people can be equally successful with different combinations of approaches. However, the fact that students tended to work almost exclusively with a narrow range of strategies, suggested that they may not have been very flexible in their choice or management of problem-solving strategies. This may be inconsistent with the findings of other research, that successful teachers use a wide variety of methods (Jacobsen, Eggen, Kauchak and Dulaney,1985) and that flexibility should be encouraged in problem solving (Mason et al., 1987, Taplin, 1994, 1995). This raises the question of whether they should be encouraged to develop more flexibility in their own choices of strategy, so they can offer a broader range of strategies to their students. If teachers are to be flexible in offering opportunities that cater for all styles of learning in the classroom, it is probably necessary to encourage them to be more eclectic in their choice of hints, so they are able to solve problems in a variety of ways. Further research will explore whether the students can be encouraged to use a wider range of strategies and if that affects their success in solving the problems.

References

Bishop, A.J. (1983). Space and geometry. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematical Concepts and Processes*, (pp.176-204). New York: Academic Press.

Borko, H., Eisenhart, M., Brown, C., Underhill, R., Jones, D. & Agard, P. (1992). Learning to teach hard mathematics: do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*, 3, 194-222.

Cooney, T. (1994). Research and teacher education: in search of common ground, *Journal for Research in Mathematics Education, 25*, 6, 608-636.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D. & Agard, P. (1993). Conceptual knowledge falls through the cracks: complexities of learning to teach mathematics for understanding, *Journal for Research in Mathematics Education, 24*, 1,8-40.

Fernandes, D. & Vale, I. (1994) Two young teachers' conceptions and practices about problem solving, In J.P. da Ponte & J.F. Matos (Eds.) *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, Vol. 2* (pp. 328-335). Lisbon, Portugal: University of Lisbon.

Gardner, H. (1983). *Frames of Mind: The Theory of Multiple Intelligences*, New York: Basic Books.

Jacobsen, D., Eggen, P., Kauchak, D & Dulaney, C. (1985). *Methods for Teaching*. Columbus, Ohio: Charles E. Merrill.

James, J. & Taplin, M. (1994). A Hypermedia-driven Computer Assisted Mathematics Remediation Package, In Thomas Ottman & Ivan Tomek (Eds.). *Educational Multimedia and Hypermedia*, Vancouver: Association for the Advancement of Computers in Education.

Jones, D. (1995). Making the transition: tensions in becoming a (better) mathematics teacher. *The Mathematics Teacher 88*, 3, 230-234.

Krutetskii, V.A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren*, Chicago: University of Chicago Press.

Lester, F. K. (1985). Methodological considerations in research on mathematical problem-solving instruction. In E.A. Silver (Ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* (pp. 41-69). Hillsdale, NJ: Lawrence Erlbaum.

Lester, F.K.Jr. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal*

Mason, J., Burton, L. & Stacey, K. (1987). *Thinking Mathematically*. Great Britain: Addison-Wesley.

Mayer, R.E.& Sims, V.K. (1994). For whom is a picture worth a thousand words: extensions of a dual-coading theory of multimedia learning. *Journal of Educational Psychology, 86*, 3, 389-401.

McLeod, D. B. (1988). Affective issues in mathematical problem solving: some theoretical considerations. *Journal for Research in Mathematics Education, 19,* 2, 134-141.

NCTM (National Council of Teachers of Mathematics Commission on Standards for School Mathematics, 1989). *Curriculum and Evaluation Standards for School Mathematics*, USA: NCTM.

Owens, K. (1994). Visual imagery employed by young students in spatial problem solving. Paper presented at the Australian Association for Research in Education Annual Conference, Newcastle.

Polya, G. (1957). *How to Solve it (2nd Ed.)*, NJ: Princeton University Press.

Resnick, Lauren B. (1987). 'Learning in school and out'. *Educational Researcher, 16*, 13-20.

Resnick, L. B. & Ford, W.F. (1981). *The Psychology of Mathematics for Instruction*, Hillsdale, NJ: Lawrence Erlbaum

Schifter, D. & Fosnot, C. (1993). *Reconstructing Mathematics Education*. NY: Teachers College Press.

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*, London: Academic Press.

Shama, G.& Dreyfus, T. (1994). Visual, algebraic and mixed strategies in visually presented linear programming problems. *Educational Studies in Mathematics, 26*, 45-70.

Swafford, J.O. (1995). 'Teacher preparation'. in I.M.Carl. (Ed.) *Prospects for School Mathematics* pp. 157-174. Reston, Virginia: NCTM.

Taplin, M. (1994). Development of a model to enhance managerial strategies in problem solving, *Mathematics Education Research Journal, 6*, 79-93.

Taplin, M. L.(1995). An exploration of persevering students' management of problem solving strategies, *Focus on Learning Problems in Mathematic, 17*, 1, 49-63.

Taplin, M.L. & James, J.S. (1994). Developing a computer assisted mathematics tutorial for preservice teachers: some pedagogical issues. In J. Willis, B. Robin & D.Willis (Eds.). *Technology and Teacher Education Annual* (pp. 524-528), Charlottesville, VA: Association for the Advancement of Computing in Education.

Thomas, N. & Mulligan, J. (1994). *Researching mathematical understanding through children's representations of number*. Paper presented at the Australian Association for Research in Education Annual Conference, Newcastle.

Watson, J.M., Collis, K.F. & Campbell, K. J. (1994). Developmental structure in the understanding of common and decimal fractions. *Focus on Learning Problems in Mathematics, 17*, 1, 2-24.

Appendix

Problems

Problem 1

An article is marked for $68. If a customer is given a discount of 12%, what is the amount paid?

Problem 2

Twelve friends had enough money to buy five Mars Bars between them. If the Mars Bars were divided evenly, how many would each person get?

Problem 3

A concrete mix is made by volume: 3 parts gravel, 2 parts sand and 1 part cement. In 12 cubic metres, what is the volume of (a) cement and (b) sand?

Problem 4

The area of a rectangle is 96 square centimetres and the perimeter is 560 millimetres. Find the length and width of this shape.

Problem 5

At my shack I need to build a rectangular tank which will hold 1000 litres of water. What proportions should the tank be so that it will require the smallest quantity of metal? (Hint: a 10cm x 10cm x 10cm cube holds 1 litre of water.)

Problem 6

Suppose a wire is stretched tightly around the Earth. (The radius of the Earth is approximately 6400 km.) If the wire is cut, its length is increased by 20m, and it is placed back around the Earth so that it is the same distance from the Earth at every point, could you walk under the wire? How far would the new wire be above the Earth?

CHILDREN'S REPRESENTATION OF THE COUNTING SEQUENCE 1-100: COGNITIVE STRUCTURAL DEVELOPMENT

Noel Thomas, Charles Sturt University, Bathurst, Australia,
Joanne Mulligan, Macquarie University, Sydney, Australia, and
Gerald A. Goldin, Rutgers University, New Brunswick, NJ, USA

This paper reports further results and analysis from a cross-sectional, empirical study of children in Grades K-6, and additional high-ability 3rd-6th graders. We focus on children's understandings of the base ten numeration system, exploring how their internal representational systems for numbers change through a period of structural development and become eventually powerful, autonomous systems.

Introduction and Theoretical Perspective

A broad study, conducted in Australia, was designed to explore the relationship between children's counting, grouping and place value knowledge, and their structural development of an internal representational system for numbers associated with the base ten numeration system. Our aim in the overall study is to achieve a detailed description of children's representational capabilities as evidenced in a wide range of numeration-related tasks (Thomas, Mulligan & Goldin, 1994; Thomas & Mulligan, 1995). In our earlier papers we analysed children's external representations of the numbers from 1-100, discussing the relationship between internal imagistic representations, dynamic imagery, and structural development of the number system. We found evidence that children's internal representations of numbers are often highly imagistic and unconventional. We inferred both static (fixed) and dynamic (changing or moving) internal representations, noting that children with high levels of understanding of numeration showed evidence of both dynamic imagery and structural development in their representations. In this report we explore further how children develop their understandings of the base ten numeration system. From their external representations we infer aspects of the period of structural development that leads ultimately to the children's construction of powerful, autonomous internal representational systems for numbers.

Examples of children's external physical, pictorial, or notational number representations have also been described in other recent studies, cited in our earlier papers. These studies raise the question of how children's representations of the number system link with their conceptual understandings of numeration, and the influence of their representational capabilities on how their knowledge is applied in problem-solving situations. Many children appear unable to represent structures related to base-ten grouping of number.

The theoretical basis for the present discussion is a model of problem-solving competency structures and their development based on systems of internal cognitive representation (Goldin, 1987, 1992; Goldin & Herscovics, 1991). In this model many internal systems of representation, of five different kinds, develop over time through three main stages: (i) an inventive/semiotic stage, in which characters and configurations in a new system are first given meaning in relation to previously-constructed representations; (ii) a structural development stage, where the previously existing system functions as a kind of template on which the new system is modeled; and (iii) an autonomous stage, where the new system of representation functions independently of its precursor, and can assume new meanings in new contexts. We propose that children's internal systems of representation of numeration go through such stages, and that as the system moves towards becoming autonomous, the external representations the child produces become more powerful, more conventional, and more capable of interpretation in a variety of new contexts.

Representational systems: We focus the discussion of numeration here on three types of internal systems of representation: verbal/syntactic (using mathematical vocabulary, developing precision of language, self-reflective descriptions); imagistic (non-verbal, non-notational mental models such as visual and kinaesthetic representations); and formal notational (using notation, relating notation to conceptual understanding, creating new notations). We further analyse the role of imagery in representation and in the construction of relational understanding in mathematics (Presmeg, 1986, 1992; Bishop, 1989; Brown & Wheatley, 1989, 1990; Mason, 1992; Brown & Presmeg, 1993).

Structural development of numerical representation: Numeration involves the development of an increasingly sophisticated counting scheme and system of notation for recording the numbers generated. Initially a system of units, the counting scheme must drive the representation of assemblages partitioned into groupings of ten. The 'ten', while retaining its signification of ten units, becomes itself an iterable unit. This in turn may be operated on by the relation 'form a group of ten', to construct the new unit of 'hundred'. Arrays provide one conceptualisation of the multiplicative process, and can illustrate recursively the relations 'multiply by 10', 'multiply by 100', etc. in the numeration system.

Types of imagery: Personal visuo-spatial representations of number (number-forms) were described long ago by Galton (1880). Seron et al. (1992) suggest that the number-form is a more accomplished development of a general disposition of people to encode numbers in a visual way. They conclude that number-forms are used to code the number sequence, and that the function (if it exists) of this phenomenon should be examined in number and calculation processing. The clinical interview studies of students' thinking cited above indicate consistently that students use imagery in the construction of mathematical meaning. As Mason suggests, images can be viewed as either eidetic (fully formed from something presented), or constructed (built up from other images). The meaning-constructing process continues as the 'mental picture' is described, drawn, compared and discussed.

Methodology

Two samples of children (for comparison purposes) were administered task-based problem-solving interviews. A cross-sectional sample of 166 children in Grades K-6 were randomly chosen from 8 State schools in the Western Region of New South Wales (NSW), representing a wide range of mathematical ability. A high-ability sample consisted of 92 children in Grades 3-6 from 84 country and city schools in NSW, selected by teachers for participation in a Gifted and Talented program; 79 of these children had been included at the time of our 1994 report.

Interview tasks: The children in the cross-sectional sample were interviewed individually in two sessions, using 25 tasks designed to explore their understandings of numeration. The children in the high-ability sample were interviewed individually once, using selected numeraton and visualization tasks. The numeration tasks addressed counting; grouping-partitioning; place value; structure of numeration; and visualization. In one of the visualization tasks children were asked to close their eyes and to imagine the numbers from one to one hundred, and then to draw the pictures that they saw in their minds. They were also asked to explain the image and their drawing. This visualization task was asked first, prior to other numeration tasks, so that responses could not be influenced by the representations used by the researcher in other tasks. The interview transcripts and the pictorial and notational recordings of all the children were analysed, looking for evidence of representations indicative of the stages described above.

Illustrative Evidence for Cognitive Structural Development

How does the external imagery produced by the child relate to his/her internal representations of structural features of the numeration system and to the construction of relational understanding? We conjecture that it is possible to infer (on a tentative basis) aspects of the child's internal imagistic representations of this structure, from the external representations observed. The resulting internal representations are fluid and changing, as evidenced by the transitory nature of imagery as learning occurs. We further suggest that a child will benefit from having available a variety of images for use in representing mathematics, so that salient features of particular imagistic representations can be drawn on in a variety of situations, and flexibility of thought developed.

The data to this point suggest that the further the representational system has developed structurally, the more coherent and well-organised will be the external representation, and the more competent will be the child numerically. Here space permits just a few illustrative cases. Figs. 1-4 provide examples of how we interpret the imagery the children produce as evidence for representational acts associated with various stages of development of their internal systems for numeration.

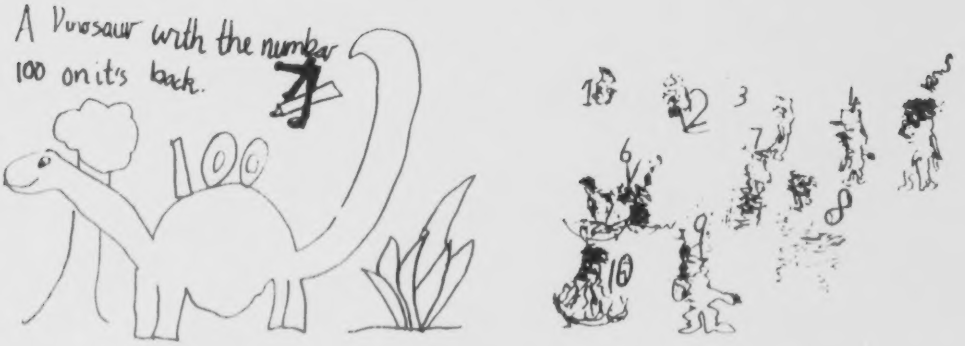

A Dinosaur with the number 100 on it's back.
100

Figure 1. Magnus (Grade 3) and Andre (Grade 3):
Evidence for the Inventive Semiotic Stage of Construction

Magnus drew a dinosaur with the number 100 on its back. This pictorial representation appears to reflect the association of the number 'one hundred' with something large, an early semiotic act. There is no indication of a counting sequence, but rather a focus on the part of the question most significant to the child; at least one aspect of the child's semantic content of 'one hundred' (size) is represented visually here. Andre drew idiosyncratic figures for each of the numbers 1 to 10, saying that "one faded, then two came - the people and animals moving around". In this pictorial representation we find evidence not only of inventive 'meaning' assigned to numerical symbols, but of an emerging awareness of sequence. (The drawing is restricted to the part of number sequence with which Andre is familiar, as evidenced by his performance on other tasks).

Fig. 2 shows Naomi's drawing of ten columns of ten circles. We infer from this ikonic representation that Naomi is developing some structure to her internal representation of the number sequence. It is a reasonable conjecture that her external representation has been driven by concrete experiences of grouping objects into tens, leading to the internal, imagistic capability of representing such groupings.

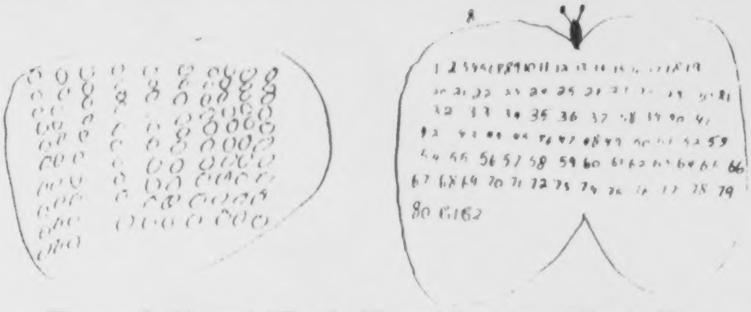


Figure 2. Naomi (Grade 3) and Summer (Grade 3)
Evidence for the Structural Development Stage of Construction

Summer's notational representation also seems to display an attempt to fit the known linear sequence into an array structure. Groupings in the rows appear to relate to Summer's notion of the prominent numbers up to twenty, and we detect some semblance of decades.

Both Naomi and Summer thus show evidence of structural development in their internal representational systems for the number sequence. Naomi represents objects as units, while Summer represents numerals; and they have used different scaffolds (groups of tens, and linear sequence) in their respective visualisations.

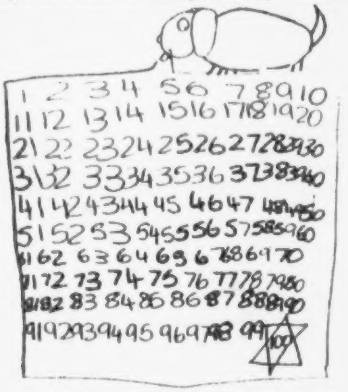

1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50
51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67 68 69 70
71 72 73 74 75 76 77 78 79 80
81 82 83 84 85 86 87 88 89 90
91 92 93 94 95 96 97 98 99
100

Figure 3. Cassie (Grade 4)

Evidence for Advanced Stage of Strucutral Development

Cassie (Fig. 3) drew an array with the number sequence in rows of ten. She could describe the notational representation as, "a hundred is ten rows of ten". We infer that Cassie's internal representation involves both the notion of sequence and the idea of groupings by ten, including iteration of that idea relating to the notational system. Edward (Fig. 4a) also showed an array structure in his spontaneous imagery for the numbers 1 to 100. When Edward was further asked to show the patterns of ten in the numbers, he too described one hundred as ten tens (Fig. 4b).

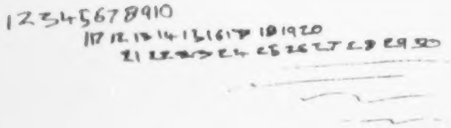

12345678910
11 12 13 14 15 16 17 18 19 20

Figure 4a. Edward (Grade 6)

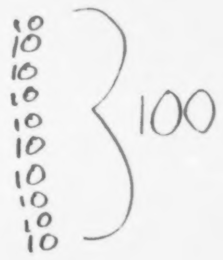

10
10
10
10
10
10
10
10
10
10
100

Figure 4b. Edward (Grade 6)

Evidence for Advanced Stage of Structural Development

BEST COPY AVAILABLE

From the children's performance on other tasks in the study, there is evidence that Cassie and Edward are able to interpret numerical representations in a variety of contexts, so that as structured systems of internal, cognitive representation, they can reasonably be considered to have reached an autonomous stage of development.

Data taken in just one or two interviews do not permit us to trace the process of construction of internal representational systems in individual children. But the variations we observe across different children strongly suggest that such systems are not fully developed at any one time, but are built up over time. Previously developed representations may serve to provide students with a framework (scaffolding, or template) on which new, meaningful representational configurations can be fit (new knowledge), and new cognitive structures built. During the many steps that occur in the structural development stage of numeration systems, we believe that the variety and meaningfulness of the images facilitate passage to an autonomous representational system of number. While the representations may be constructed in response to specific tasks, conceptual understanding of numeration must involve many experiences with the representation of numerical ideas, across many different tasks, with meaningful semantic relationships among them.

Imagery and the Learning Process

In our studies it appears that the active processing of images plays an important part in the development of the child's understanding of numeration. Since images are built up from words, notations, and other images, the representations do not become autonomous until the idea makes sense. That is, numeration can be used itself as a tool in mental thinking, flexibly and independently of any particular image. To facilitate this children's mental images should be described, drawn, compared and discussed. As their internal structures are developing, the children's external representations, both static and dynamic, may not correspond to conventional mathematics, or be uniform in nature from one child to the next. They should be expected to reflect each child's unique internal constructions at that time. Such a range of available images is, in our view, healthy; the images are constructed so that an internal representation system that 'works' can be built up. Thus the teaching/learning situation needs to provide opportunities for children to develop and represent structurally meaningful mathematics.

Conclusions and Limitations

This is a descriptive study. Although it was conducted with a large number of children, it was not designed or intended as a controlled experiment permitting immediate generalization. Our methods of inferring aspects of children's internal representations from their externally produced representations are still exploratory, and not yet subject to tests of validity or inter-researcher reliability. And as noted, it is not a longitudinal investigation.

With these limitations, we believe we continue to find evidence that cognitive representational systems develop over time. We find behaviors from which representational acts can be inferred: inventive-semiotic acts of initially assigning imagistic meanings to or identifying them with mathematical words and symbol-configurations; structural developmental acts associated with sequences of numbers, groupings by tens, recursive grouping, and other mathematical structures; and autonomous acts in which insightful, mathematical meanings are freely and flexibly found in other systems of representation, distinct from those used initially in constructing the numeration system.

A longitudinal study of the mathematical development of 22 children in Grades 3-6 in New Jersey has been conducted at Rutgers University. Task-based interview data were gathered during 1992-94, and are presently being analyzed (Goldin et al., 1993). A longitudinal study at Macquarie University of 120 Australian children in Grades 2-3, commencing in 1996, will focus on children's construction of numerical relationships (Mulligan, Mitchelmore and Outhred, in progress). We anticipate that these studies will shed further light on the processes whereby children's internal systems of representation develop, and how such processes can be inferred from task-based interviews with children.

References

Bishop, A. J. (1989). Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics*,11, 1, 7-15.

Brown, D. L. & Presmeg, N. C. (1993). Types of imagery used by elementary and secondary school students in mathematical reasoning. In Hirabayashi, N. Nohda, K. Shigematsu and F.-L. Lin (Eds.), *Proceedings of the 17th Annual Conference of PME,* Vol. II. Univ. of Tsukuba, Japan: Program committee, 137-144.

Brown D. L. & Wheatley, G. H. (1989). Relationship between spatial ability and mathematical knowledge. In C. A. Maher, G. A. Goldin, and R. B. Davis (Eds.), *Proceedings of the 11th Annual Meeting of PME-NA*, New Brunswick, NJ, Rutgers Univ. Center for Mathematics, Science and Computer Educ., 143-148.

Brown D. L. & Wheatley, G. H. (1990). The role of imagery in mathematical reasoning. In G. Booker, P. Cobb & T. de Mendicutti (Eds.), *Proceedings of the 14th Annual Meeting of PME*, Oaxtapec, Mexico: Program committee, 217.

Galton, F. (1880). Visualised numerals. *Nature*, 21, 252-256 and 494-495.

Goldin, G. A. (1987). Cognitive representational systems for mathematical problem solving. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics.* Hillsdale, NJ: Erlbaum, 125-145;

Goldin, G. A. (1992). On developing a unified model for the psychology of mathematical learning and problem solving. In W. Geeslin and K. Graham

(Eds.), *Procs. of the 16th Annual Conference of PME,* Vol. 3. Durham, NH, USA: Univ. of New Hampshire Dept. of Mathematics, 235-261.

G. A. Goldin, V. A. DeBellis, A. M. DeWindt-King, C. B. Passantino, and R. Zang (1993). Task-based interviews for a longitudinal study of children's mathematical development. In I. Hirabayashi, N. Nihda, K. Shigematsu, and F.-L. Lin (Eds.), *Proceedings of the 17th Annual Conference of PME,* Vol. 1. University of Tsukuba, Japan: Program committee, 197-203.

Goldin, G. A. & Herscovics, N. (1991). The conceptual-representational analysis of children's early arithmetic, In R. G. Underhill (Ed.), *Proceedings of the 13th Annual Meeting of PME-NA,* Vol. 1. Blacksburg, Va.: Virginia Tech. Division of Curriculum & Instruction, 118-124.

Hiebert, J. & Wearne, D. (1992). Links between teaching and learning place value with understanding in first grade. *Journal for Research in Mathematics Education,* 23 (2), 98-122.

Mason, J. (1992). Doing and construing mathematics in screen-space. In B. Southwell, B. Perry, and K. Owens (Eds.), *Proceedings of the 15th Annual Conference of the Mathematics Education Research Group of Australasia,* Nepean, Australia: University of Western Sydney, 1-17.

Presmeg, N. C. (1986). Visualization in High School Mathematics. *For the Learning of Mathematics,* 6(3), 42-46.

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics* , 23, 595-610.

Seron, X., Pesenti, M., Noel, M. P., Deloche, G. & Cornet, J. (1992). Images of number, or "When 98 is upper left and 6 sky blue". In S. Dehaene (Ed.), *Numerical Cognition,* Cambridge, MA: Blackwell, 159-197.

Thomas, N., Mulligan, J. & Goldin, G. A. (1994). Children's representation of the counting sequence 1-100: Study and theoretical interpretation. In J. Matos (Ed.), *Proceedings.of the 18th Annual Conference of PME,* Vol. II. Univ. of Lisbon, Portugal: Department of Education, Faculty of Sciences, 1-8.

Thomas, N. & Mulligan, J. (1995). Dynamic imagery in children's representations of number. *Mathematics Education Research Journal*, 7, 1, 5-25.

COLLEGE STUDENTS' CONCEPTIONS OF VARIABLE

***Maria Trigueros**, ITAM; Sonia Ursini, CINVESTAV-IPN; Araceli Reyes,ITAM.*

México

ABSTRACT

This paper presents the results of a study in which college student's conceptions of variable were investigated. A decomposition of variable was the basis for the design of a questionnaire and the qualitative and quantitative analysis of the responses. The results obtained show that there is a permanence of student's difficulties to deal with variable and that their proficiency in conceptualising and handling the different aspects of it is far from what should be desired.

Introduction

Advanced mathematical thinking requires a good understanding of algebra concepts, particularly that of variable. The processes leading to its construction have been studied observing and analysing secondary school pupils' work (2), (4), (8). The difficulties students have with its different manifestations (general number, specific unknown, variables in functional relationship) have been highlighted and the most common errors have been stressed (1), (3), (5). Different teaching approaches have been presented that implicitly suggest a hierarchical order for the acquisition of the concept of variable (2), (4). Few research, however, has been done to study college students' conceptions of variable.

It is to be expected that after several high school algebra courses the concept of variable has been learnt, that is, it is supposed that students should have a fluid syntactic handling of variable, should be able to distinguish its different manifestations, and to handle them in a flexible and integrated way. To gain a deeper understanding of how students learn this concept and of the didactic phenomenology associated with it, it is important to complement research already done with secondary school children, with research that looks carefully at more advanced students conceptions and capability to handle this concept. Our research project aims at contribitung to fill this gap.

In two previous reports (6), (7) we presented the first phases of the study. We reported there some preliminary results about college students difficulties to interpret the different variable aspects. We highlighted two important facts: 1) Several students perceptions of variable remain unchanged along different school levels and 2) Mexican college students have a tendency to interpret the variable as a specific unknown independently of the nature of the particular problem posed. These results led us to refine our instruments in order to get a deeper understanding about college students capability to discriminate and shift between different uses of variable, and to make explicit connections between them, as well as to compare their conceptions with secondary school pupils in order to see if there is a significative progress in the way variable concept is conceived. To accomplish this goal we constructed a framework which was used for designing the experiment and for analysing the data obtained.

In this paper we present a decomposition of the concept of variable in which one basic assumption is that these different aspects have the same cognitive status . We show how we used the decomposition in the design of a questionnaire and in the analysis of students' responses. We comment on some of the characteristics of the test itself, and present the results of both a quantitative and a qualitative analysis of the data obtained. We end up with some comments on the didactic phenomenology of the concept of variable and on the possibility to use the results of this study to design a teaching strategy that may enable students to deal with it as an integrated mathematical object.

Theoretical Framework

Our perspective is partially based on the framework proposed by Dubinsky for the construction of mathematical concepts (9). Dubinsky states that in order to understand the way students learn mathematics it is necessary to analyse the different concepts involved to isolate their main components and to give explicit descriptions of possible relations between them. The product of this analysis is called a genetic decomposition of the concept. Genetic decompositions can be a useful tool to start research on students understanding of mathematics and can serve as a guide in the design of teaching strategies.

Our theoretical framework starts with a decomposition of the concept of variable shown on the following table which is based, on the one hand on a thorough study of the concept of variable, and on the other hand on already known student's difficulties when dealing with this concept. In contrast to a genetic decomposition, the decomposition of variable we present highlights only those aspects that seem to be relevant for its construction from the point of view of an expert and considers that all of them have the same hierarchy. It is important to remark that we do not intend to establish stages for the learning of this concept, we rather pretend to point out different features that can be constructed by the students in a non linear way and that are important in order to get an integrated view of the variable concept.

As any model, this one can be further refined and has no pretension of being unique. It is supposed, however, to serve as a useful tool in the analysis of the data.

In our decomposition we start from the point of view that the concept of variable appears in algebra through different facets, for example specific unknown, general number and variables in functional relationship. These have been shown to be the most relevant through the literature and in school teaching practices. To construct the concept of variable implies the capability of integrating all these different aspects and the possibility of passing from one to another in a flexible way. Furthermore we consider that in order to work with each one of these aspects it is necessary to be able to conceptualise, symbolise, interpret and manipulate them.

The decomposition was used to design a questionnaire in which each item relates to one particular aspect of variable. It was also used for interpreting students' responses in order to obtain a global view of their conception of variable. Moreover, considering that students conceptions are related with the meaning they are getting for the concept through the school system, we can also get from the analysis of the data a picture of the main characteristics of the Mexican school system.

DECOMPOSITION OF VARIABLE

	Conceptualisation and Symbolisation	**Interpretation**	**Manipulation**
Generalised Number	Conceptualisation of a general object involved in general methods or rules deduced from numeric and/or geometric patterns and families of similar problems; and its symbolisation.	Interpretation of a symbol as a general object in algebraic expressions or in general methods.	Factorise, simplify and expand to rearrange expressions.
Specific Unknown	Conceptualisation of an unknown in a particular situation and/or in an equation and its symbolisation .	Interpretation of a symbol as a specific unknown in equations in which it appears once or more times.	Factorise, simplify, expand, transpose or balance an equation to make a variable the subject of equation.
Variable in a Functional Relationship	Conceptualisation or symbolisation of functional relationships starting from a table or graph or a problem in natural language.	Interpretation of correspondence and joint variation in analytical expressions, tables and graphs.	Factorise, simplify, expand to rearrange an expression, substitute values to determine intervals of variation, maximum/minimum values and global behaviour of the relationship.

Methodology

The questionnaire consisted of 65 open ended simple items each one isolating as best as possible one of the aspects of the variable decomposition (it will be available during the presentation). It was applied to 164 Mexican starting college students. The responses were analysed quantitatively and qualitatively. The quantitative analysis aimed at validating the instrument and getting an overview of students' capability for working with different uses of variable. The qualitative analysis aimed at pinpointing students' ways of conceptualising, interpreting, symbolising and manipulating different uses of variable in order to obtain a deeper understanding of their variable conception and to see if the difficulties secondary school pupils usually have, have been overcome.

Quantitative Analysis of the results

The quantitative analysis was made in terms of frequencies of responses and in terms of Classical Test Theory. After analysing the questionnaire as a whole, the items were regrouped in three subtests depending on the particular aspect of variable tackled and each subtest was analysed separately.

The mean for the global scores for the questionnaire was 33.33, with a standard deviation of 9.79. The distribution was very near normal with a skew of 0.16 which indicates abundance of scores in the lower part of the distribution. The reliability coefficient of 0.89 indicates a good internal consistency of the questionnaire.

The analysis of the responses to the three subtests can be summarised as follows:

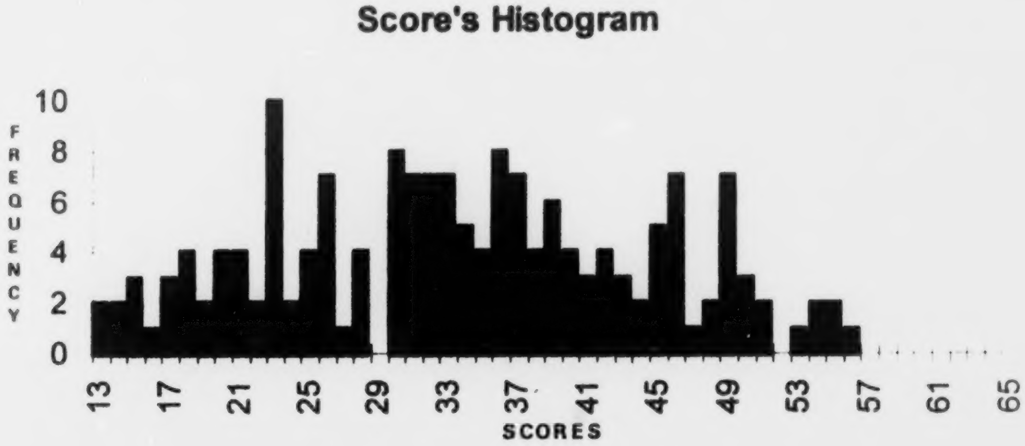
	Specific Unknown	Generalised Number	Var. in Funct Relationship
number of questions	19	19	27
mean	10.89	9.66	12.77
std.	2.95	3.9	4.55
skew	0.355	-0.076	0.068
reliability	0.73	0.77	0.79

These results show that the reliability is good for the three subtests and that the variance is larger for items dealing with variables in functional relationship. The distribution for the specific unknown subtest is slanted. This indicates that students had fewer difficulties with the items in this subtest. This result is confirmed by its lower standard deviation.

Global scores are lower than we might expect for college students who rate well in other kinds of problematic situations. A graph of the frequency of scores obtained by the students is shown in figure 2. Moreover, only six questions were answered correctly by more than 90% of the subjects. Four of these required to recognise simple numeric or geometric patterns and to show explicitly the following step. The other two required symbolising variable as specific unknown in linear equations (e.g. "Write a formula that expresses that an unknown multiplied by 13 is equal to 137"). The percentage of correct answers decreased substantially (3.7% correct answers) for items involving pattern recognition when the expression of the general subsequent step was required (e.g. to symbolise the number of elements added to a given pattern in order to go from step n to step n+1).

All the problems requiring interpretation of variable as specific unknown in simple quadratic expressions, like for example $(x+3)^2=36$, were answered correctly by less than 10% of the students. Moreover, only 7.3% of the students could write a quadratic equation for solving a word problem.

Items involving the deduction of a rule of correspondence from the analysis of a tabular record of a functional relationship or the description of a dependent variable in terms of the variation of the independent one were answered correctly by only 11% of the students. Students' greater difficulties were related to questions dealing with intervals of variation. Only 1.7% of the students could identify the range of variation of the dependent variable given an interval of variation for the dependent one, when the functional relationship was presented in tabular or graphic form..



The quantitative analysis put forward two striking results: none of the tested students could answer correctly the whole questionnaire, and not a single question was answered correctly by all the students even though all the items of the questionnaire were simpler than those that use to appear in standard school tests and that all the students tested rated high on an exam which is equivalent to SAT (Standardised Academic Test used as entrance examination in USA universities). It is worth reminding that in Mexico all students have had at least four mandatory algebra courses before starting college studies.

Given the students capabilities to solve problematic situations in other contexts different to algebra, this global perspective suggests that high school algebra courses are not fulfilling their goal. There is no evidence in the results obtained of significative conceptual changes in terms of the handling of the concept of variable.

Qualitative Analysis of the results

Even if The quantitative analysis of the responses to the questionnaire suggests that there are not significant differences between college and secondary students, a qualitative analysis shows that there is some improvement in their capability to interpret, symbolise and manipulate all the aspects of variable when they face simple expressions, but the difficulties reappear as soon as their complexity grows.

Overall qualitative results contradict the expectations of proficiency in the capability to handle of the variable concept for college students, and can be summarised in terms of the decomposition proposed as follows:

The majority of students could not determine intervals either from a given analytical expression of a relationship nor from its tabular or graphic representation. Abundance of incorrect responses to items asking about the behaviour of one of the variables when the other variable changes was also observed. For example to an item in which a graph was shown, and students were asked to determine the interval in which the dependent variable decreased, most of the students confounded decreasing behaviour with negative values. This suggests an improper interpretation of the global behaviour of the relationship. To another item asking for the values of x corresponding to the values of y in the interval (3,10) if $y=x+3$, many of the students gave only a few positive integers as a response. This evidences a lack of capability to handle and interpret correspondence and variation.

A tendency to use the equal sign as a linking symbol to connect subsequent steps in the construction of a correspondence, instead of using it as an equivalence indicator was also observed. For example to a problem in which they were asked to find the relationship between the weight and the displacement of an indicator in a balance, the answer produced by about a half of the students was 1kg=4cm; and to an item asking to analyse a table presenting a time-velocity relationship, the majority wrote 10sec=30m/sec. This suggests that students need an objective way to appreciate the actual relationships and that they use the equal sign as a tool to analyse the problem in the search for the correspondence. This seems to be an indicator of students necessity to perform direct actions on the data involved, instead of being able to handle them in a more abstract way in order to interpret the correspondence and joint variation in the expressions.

College students and algebra beginners seem to share the same conception of general number: This is suggested in the following examples:

College students can translate from natural language to algebraic language when chains of operations are not needed (e.g. "Write a formula which means: 4 added to n+5"). When chaining of operations is needed, they tend to name the possible result of each of its parts before continuing. For example. to answer the item "Write an expression that says: an unknown number divided by 5 and the result added to 7" the great majority of students wrote $x/5=y+7$. This suggests a weakness in the understanding of the variable as a general number and a need for defining a new object emanating from the result in order to continue the operations.

The same weakness to interpret the generalised object manifests itself again in the difficulties students have in using a given symbol to write a known formula. For example, when asked to write a formula for the perimeter of a square of side a, most of them wrote P=4l, that is, a formula learnt by heart at school. Similar responses are given by algebra beginners (10).

There is a tendency to interpret the variable in any algebraic expression as a specific unknown. For instance to an item asking for the values that a can take in the expression $7+a+a+a+10$ more than half of the students made the expression equal to zero and solved it. This also indicates a difficulty for interpreting a general object.

Even though students can manipulate, interpret and symbolise the variable as a specific unknown in simple problems, they have a tendency, similar to that of algebra beginners, to solve problems through arithmetic procedures. In a very simple age problem asking to symbolise the equation needed to solve it, the great majority of students avoided symbolisation and guessed the numeric answer. If the problem is difficult enough as to make an arithmetic approach impossible, students have difficulties in identifying the unknown and posing an equation.

An avoidance to perform algebraic manipulation was also found, even though manipulation techniques are strongly emphasised in Mexican curriculum. Again we find here another symptom of the lack of comprehension of variables.

A possible explanation for the persistence of students difficulties can be that current teaching high school practices reinforce each one of the different aspects of variable separately and in a hierarchical way, stressing their specific manipulation and transformation rules. However, different uses of variable share the same symbolism, syntax and manipulative rules. This fact should be taken explicitly into account in order to avoid possible confusions and misunderstandings.

It has been shown that algebra beginners can deal indistinctly with any of the aspects of variable. It could be that most of the students' misunderstandings derive from the fact that there is no effort whatsoever tending to help pupils to distinguish between the different facets of variable and to integrate them as parts of the same conceptual entity.

Conclusions

This work proves that research on college students' conceptions is important and effective in highlighting some aspects of the learning process of the concept of variable.

One important finding of this study is that although there are some improvements in the way college students handle variables, there is a persistence of approaches found for algebra beginners. College students have not a fluid enough syntactic handling of variable and are nor able to distinguish between its different uses. This suggest that there is not enough emphasis in conceptualisation and so apparent improvement in the ways students handle variables might be due to the standard drill and practice activities and mechanical application of algorithms and memorised rules.

A fluid syntactic handling of variable requires an integral view of this concept, as well as the capability to distinguish between its different uses. The level of conceptual development of the subjects of the study, given their age, does not justify the lack of conceptualisation of variable observed. This suggests a deficient highschool teaching that does not emphasise conceptualisation.

A synchronised teaching of all the aspects of variable may help in the construction of this concept as a whole as well as facilitate the distinction of its different uses when needed. The decomposition of variable presented in this paper allowed to identify some important local weaknesses in college students' conceptualisation of variable, and suggests that a decomposition like this one may serve as a useful guide in the design of specific teaching strategies.

References

[1] **BOOTH L.** (1984), Algebra: Children's Strategies and Errors, *NFER-NELSON, Windsor.*

[2] **HART, K.** (1981), Children's understanding of Mathematics: 11-16. *Alden Press. Oxford London and Northampton. Great Britain.*

[3] **KIERAN C.** (1984), Cognitive Mechanisms Underlying the Equation - solving Errors of Algebra Novices, *Proceedings of the Eighth PME Conference,* **pp. 70-77.**

[4] **KUCHEMANN D.** (1980), The Understanding of Generalised Arithmetic (Algebra) by Secondary School Children, *PhD Thesis, University of London.*

[5] **MARKOVITS Z., EYLON B.S. and BRUCKHEIMER M.** (1988), Difficulties Students Have with the Function Concept, *in Coxford A.F. and Shulte A.P. (Eds.), The Ideas of Algebra K-12,* **pp. 43-60.**

[6] **QUINTERO, R; REYES, A.; TRIGUEROS, M.; URSINI, S.** (1994) Misconceptions of variable *in Proceedings of the Sixteenth Annual Meeting. North American Chapter of the International Group for the PME.* **p. I-185.**

[7] **QUINTERO, R; REYES, A.; TRIGUEROS, M.; URSINI, S.** (1995). Students' approaches to different uses of variable. *Proceedings of the Nineteenth PME Conference.*

[8] **REGGIANI, M. (1994) Generalisation as a basis for Algebraic Thinking: Observations with 11-12 Year Old Pupils. *Proceedings o of the Eighteenth International Conference for the PME.* pp. IV 97-104.**

[9] **TALL, D.**(1991) Advanced Mathematical Thinking. *Kluwer Academic Publishers. Mathematics Education Library.*

[10] **URSINI S.** (1994), Pupils' Approaches to Different Characterisations of Variable in Logo, *PhD Thesis, University of London Institute of Education*

SEEING IS REALITY : HOW GRAPHIC CALCULATORS MAY INFLUENCE THE CONCEPTUALISATION OF LIMITS

L. Trouche, D. Guin

ERES, Université Montpellier 2, FRANCE.

Abstract : Contrary to an accepted opinion - graphic calculators are tools for teaching - we point out that they are continuously used by students, but not very taken into account by the french educational system. However, graphic calculators have a strong influence on conceptualisation of the fundamental notion of limit. This influence is particularly strong because there is no explicit definition of limit in french secondary mathematics curriculum. Without definition, the image - of the calculator - is the master. Therefore, the use of a graphic calculator may induce procedures linked to more primitive conceptions of limits. To tackle this problem requires a change in curriculum and a reflective integration of calculators in mathematics courses.

I Introduction

In [Espinoza & al 95] one can find an analysis about the concept of limit in the spanish secondary educational system which points out activities - highly valued in cognitive education researches - which are *lacking* in the teaching system (lack of elementary reading technique to read a graph, no relationship between continuous and discrete,...). These results are nowadays completely also avalaible for the french educational system. Morever, about infinitesimal calculus teaching, various questions arise in France. Curriculum was strongly modified : any definition of limit (even unformalized) has completely *disappeared* from secondary teaching system, even though various exercises require to handle this notion. There is *no assessment* concerning the outcomes of these changes [Artigue 93]. The official comments attached to curricula lay stress on the elaboration of *experimental processes*, in which calculus tools should play a significant role. As graphic calculators are allowed in secondary examinations, therefore students continuously use them, even though they are *not really taken into account by teachers* in classrooms. In these conditions, how students elaborate their own knowledge ?

II Objective

This study focuses on the influence of graphic calculators on the students *conceptualisations of limits* in scientific classrooms (high school level). The main hypothesis is that graphic calculators may *induce illusions* on their possibilities relative to the two types of *infinity* : the indefinitly extension and the infinite number of points in an interval [Garançon et al 93]. Indeed, results of fig 1 and fig 2 point out the fact that calculators are in practice *ignored* by teachers in the french educational system [Trouche 92] :

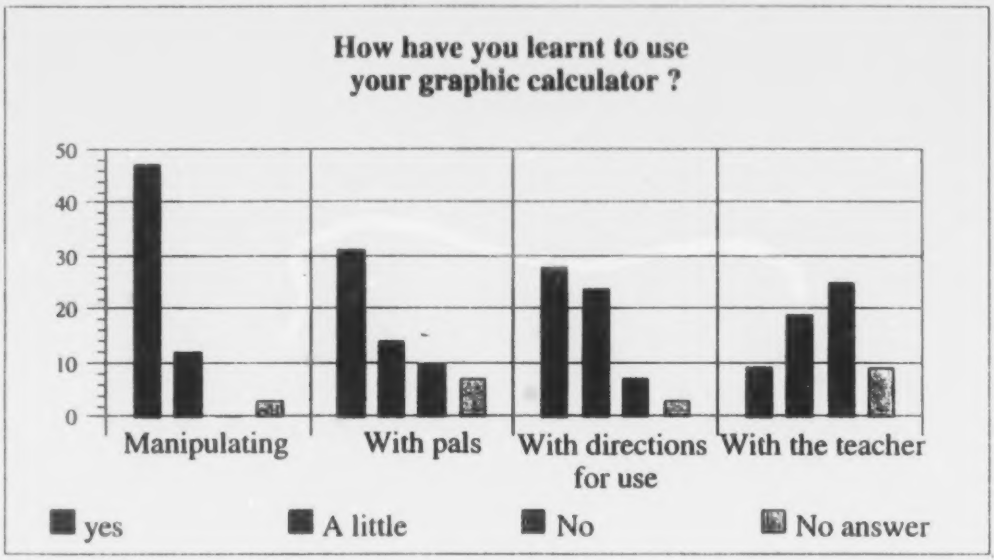


Fig. 1 (70 students 16 / 17- year - old)

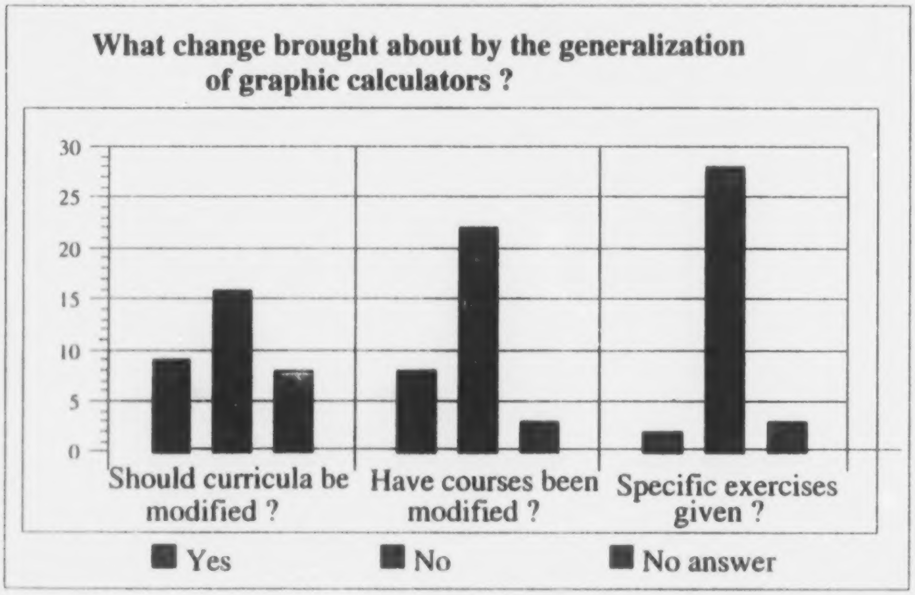


Fig. 2 (33 teachers)

At the moment when students discover infinitesimal calculus, they think that their graphic calculator can *"show"* the two types of infinity. Which implications on students'conceptions ?

III Theoretical framework

One one hand, various studies relative to the difficulties in teaching the concept of limit without calculators were carried out : epistemological obstacles [Sierpinska 85], obstacles and conceptions [Cornu 92], and models expressed by students about the convergence of numerical sequences [Robert 82]. In [Tall & Vinner 81], the research deals with the conceptualisation of limit. On the other hand, other studies [Hillel 95, Artigue 95] focus on problems - which often are also available for graphical calculators - linked to the use of Computer Algebra Systems, but not specifically about the concept of limit. Recently, in [Monaghan & al 94], there was a report on limit conceptions of students who have learnt calculus with the aid of a Computer Algebra System.

Whe are aiming to begin an anologous study with graphical calculators, which are not at all assumed by french teachers, therefore the context is completely modified. E. Goldenberg [Goldenberg 87] has already noticed that "students often *misinterpreted* what they saw in graphic representations of functions. *Left alone* to experiment, they could induce rules that were *wrong*... How do misconceptions *distort* the information that students glean from the graph?". In Goldenbergs' report, a classification was begun concerning the sources of confusions in the perception and interpretation of graphs.

Our study concerns the limit of real functions *at infinity.* We have elaborated a classification taking into account results previously recalled for analysing the link between conceptions, students procedures and their use of graphic calculators. We have pointed out *four conceptions* of a limit :

* a *"primitive"* conception : a function whose limit is infinite is a function which takes large values for large values of the variable (properties noticed for example in the case of power functions).

* a *"monotonic"* conception : a function whose limit is infinite is an increasing function (or increasing function after some value of the variable) : "To become very large, it must always increase..."

* a *"no - upper - bounded"* conception : a function whose limit is infinite is a function which may take values greater than every fixed real. This conception frequently encountered in the first year of university is linked to the confusion between necessary and sufficient condition : a function whose limit is infinite has no *"ceiling"* (but a function which has no *"ceiling"* has not necessarily an infinite limit...).

* Finally, the *"expert's"* conception that depends on a real understanding of the mathematical definition.

It is clear that these "conceptions" are *models* of appropriate students behaviours, then some behaviours may be interpreted in terms of *intermediate* conceptions.

IV Methodology

Our *hypothesis* is that *no mastered* manipulations of the graphic calculator will lead to *back up* the two first conceptions described above:

* The *primitive* conception : it is attractive and easy with a calculator to seek various values of f(x) when x is large, the calculus is immediate, then the limit of the function is really +∞ . A function whose limit is +∞ when x tends towards +∞ is a function which takes large values for x large.

* The *monotonic* conception : the screen of the calculator has a highly productive character. What is out of the screen seems to behave as a *continuation* of the screen itself. As [Garançon et al 93], "we observed an initial tendency to rely on the appearance of graphs and to *extrapolate from what was visible* ". Then the study of the function on a "*large enough* " interval will give students an idea of the global behaviour of the function. The global aspect of the curve will be supposed to give, or induce, the limit at the boundaries of the screen. Then, in this case, the study of the limit of a function amounts to the *study of its variations*. For proving that a function has +∞ as limit, it is really sufficient to prove that it is an increasing function...

To valid this hypothesis, a questionnaire was designed to elicit students conceptions of limit of a function at infinity. We have chosen students in last year of secondary school (100 students 17 / 18 year - old) and in first year of university (100 students 18 / 19 year - old) because it is the first moment when they learn a *formal definition* of the limit : it is interesting to observe the impact of this fact on the identified conceptions. Only half part of these students had the possibility to use calculators. We report on a part of this questionnaire :

Questions about definitions

Q_1 : What means $\lim_{x \to +\infty} f(x) = +\infty$?

Q_2 : How can you explain it to a younger student ?

Questions about procedures

Q_3 : Which methods do you know to prove that a function has +∞ as limit when the variable tends towards infinity ?

Questions about exercises on limits with an explicitation of used procedures

Q_4 : Have the following functions a limit at infinity ?

☐ yes ☐ no ☐ I don't know

Justify theoretically your answer.

Illustrate it with a fitted diagram.

(There was about ten "problematic" functions as, for example, ln x + 10 sin x).

V Preliminary Results

The analysis of responses enables us to valid the previously displayed conceptions. We choose answers relative to the question Q_2 which are supposed to faithfully translate the intimate students' conceptions :

	Secondary school	**University**
No answer	11	16
Tautology	0	8
Primitive conception	30	29
Monotonic conception	52	27
No - upper - bounded conception	7	14
Expert conception	0	6

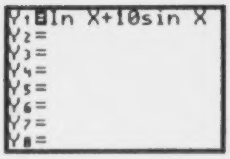
Fig. 3 (rate per cent)

There is an *improvement* between secondary school and university which still raises various questions (students at university know the expert's definition, but they not know how to *translate* it for a younger student, then they come back to a *primitive* definition). We have likewise found this improvement in the described procedures : we notice more procedures linked to variations of functions among monotonic conceptions, and more procedures linked to strategy of finding a *lower* function among no - upper - bounded conceptions.

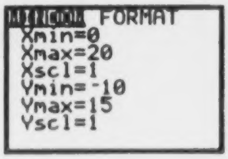
Deeper differences will appear *with the use of calculators* in the study of the limit at infinity of , for example, ln x + 10 sin x. Here is the share-out of the answers "no" :

	With calculator	Without calculator
Secondary school	25	0
University	20	0

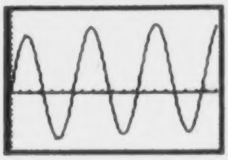
There is a real effect of calculators on these answers. Clearly, the graph of the function produced by the calculator is rather *disturbing* for a "novice " :



```
Y1=ln X+10sin X
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=
```



```
WINDOW FORMAT
 Xmin=0
 Xmax=20
 Xscl=1
 Ymin=-10
 Ymax=15
 Yscl=1
```



However, the effects of calculators are nor restricted to these points. There is a significant difference between procedures used with (or without) calculators :

* *with calculator* : students usually refer to *variations* of correspondent functions, with ambiguous sentences as " the function ln increases faster that the function sinus " ;

* *without calculator* : students more often refer to strategies based on a function *lower* than the given function to prove the result (at the two studied levels).

We simultaneously observe differences in *gestures* made in the course of the action according to the possibility to use a calculator or not :

* *with calculator* : students begin to watch the graphic representation provided by the screen. They initiate successive zooms as the photographer of *Blow up* [Antonioni 86] trying to extract from the negative the required information... They attempt to deduce from the *"profile"* of the curve informations about its behaviour towards infinity. We think at the *obstacle of first information* [Bachelard 38]. Some students proceed in an analogous manner for the function $x^2 + \sin x$. The observation of the screen points out that it is an increasing function on R^+. Then, they will prove it by derivation, convinced that they will have established that the limit is $+\infty$.

**without calculator* : students work on the algebraic form, trying to reduce it at known expressions by factorization, or to compare it to a *lower* function whose limit is infinite. Then the reference is here the course.

From this example, we better understand the *gap* between procedures used by students with calculators (or not). Without calculators, references are more actual, inserted in a course context of recently acquired knowledge. With calculators, references are deeper limited to personal experiences of the student, to images which already exist for a long time.

In this way, the use of calculators induces to *reinforce primitive conceptions* about the limit notion. However, one have to notice that after a more elaborated analysis, the situation is more complicated : some students use calculators just as an *auxiliary* tool, or as a last resort. Other students know how to *combine* different available sources of information (course, calculator, theoretical calculus). But for the majority of students, the graphic calculator is the first and often the *more influent*, sometimes the *only way* of investigation.

VI Discussion : instructional implications

Is it possible to *avoid* these negative aspects in the influence of calculators ?

There is an *unavoidable gap* between "real" mathematics and their image provided by calculators (as well at the graphic level than at the numerical one). For a calculator every number has a successor, on a graphic screen one can move from point to point. There is an unquestionable *discretisation* of the continuum, rather disturbing for the student. The social role of images has been analysed in [Debray 92]: images of screens are *more* than a representation of the reality (rather a *display* of it) : they acquire an *autonomy* in comparison with the corresponding algebraic form. This *"graphic presentation"* is supposed to *summarize* all the properties of the function, and therefore must give answers relative to infinity. However, only variations or rather the amplitude of variations can be watched : *feeling becomes proof...*

For all these reasons, teachers must *hold a place* in these process, they have to make explicit instructional interventions taking upon themselves an important responsibility of *"image"* education. It means that first they have to *integrate* graphic calculators as tool of course and to organize, when it is possible, backward and forward motions between calculators, theoretical results, and calculus by hand. Then, there are implications in the teachers' behaviour (to organize the comparison between the blackboard and the screen with a overhead-calculator, setting specific problem situations to facilitate some reorganization of the work) and in providing exercises which have to integrate tools of calculus as an *aid to conjecture, to solve and to check.* As in the experiment with graphical calculators described in [Resnick & al 94], students will have the possibility to construct very rich representations and to use it in a highly flexible manner in problem solving.

On the other hand, one have to lay stress on the fact that it is *dangerous* to introduce some kinds of "UFO" in curricula as limit of functions which *will be defined further* at university. We have pointed out an *obstacle of a didactic nature* (that is linked to the choices and charateristics of the educational system) : we cannot *avoid* that these objects *will take sense* from the available tools as graphic calculators.

VII Perspectives

It is utopian to think that a technical progress would allow us to solve problems brought about the integration of calculus tools in the mathematical course. For example, the new generation of calculators (as TI 92) has a Computer Algebra System and offers the possibility to make exact calculus, to seek limits of functions (the keyboard has a key ∞...). Certainly, students will not be mistaken on limit calculus. But will they have understood what is a function with such a behaviour towards the infinity ? What conceptions will go stronger ? What new obstacles will be created ? The problem is just shifted... How will evolve students' behaviour in front of these new materials ? We carry on this study with an experiment in a classroom where students have TI 92 at one's disposal.

References

Artigue M, 1993 : " Enseignement de l'analyse et fonctions de référence", *Repères*, vol 11, 115 - 139.

Artigue, M, 1995 : " Une approche didactique de l'intégration des EIAO ", in Guin D., Nicaud J.-F., Py D. (ed.), *Environnements Interactifs d'Apprentissage avec Ordinateur*, 17 - 29, Eyrolles, Paris.

Bachelard, G, 1938 : *La formation de l'esprit scientifique*, Librairie philosophique G. Vrin, Paris.

Cornu, B, 1992 : "Limits", in Tall D. (ed.) *Advanced Mathematical Thinking*, 153 - 166, Dordrecht : Kluwer.

Debray, R, 1992 : *Vie et mort de l'image, une histoire du regard en Occident*, Gallimard, Paris.

Espinoza, L ; Azcarate, C, 1995 : " A study on the Secondary Teaching system about the concept of limit ", *Proceedings of PME XIX*, vol 2, 11 - 17, Recife.

Garançon, M ; Kieran, C ; Boileau, A, 1993 : " Using a discrete computer graphing environment in algebra problem solving : notions of infinity / Continuity ", *Proceedings of PME XVII*, vol 2, 25 - 32, Tsubuka (Japan).

Goldenberg, E, 1987 : "Believing is seeing : how preconceptions influence the perception of graphs", *Proceedings of PME XI*, vol 1, 197 - 204, Montreal.

Hillel, J, 1995, Computer Algebra Systems as Cognitive Technologies : Implication for the Practise of Mathematics Education, Nato, Serie F, Vol 131, 18 - 48, Springer-Verlag.

Monaghan, J ; Shyashiow, S; Tall, D, 1994 : "Construction of the Limit Concept with a Computer Algebra System", *Proceedings of PME XVIII*, vol 4, 279 - 286, Lisbon.

Resnick, Z ; Schwartz, B; Hershkowitz, R, 1994 : "Global thinking "between and within" function representation in a dynamic interactive medium", *Proceedings of PME XVIII*, 225 - 232, Lisbon.

Robert, A, 1982, *L'acquisition de la notion de convergence des suites numériques dans l'enseignement supérieur*, Doctorat d'état, Université Paris VII.

Sierpinska, A, 1985 : " Obstacles épistémologiques relatifs à la notion de limite", *Revue en Didactique des Mathématiques*, vol 6.1, 5 - 67, La Pensée Sauvage.

Tall, D ; Vinner , S. , 1981 : "Concept image and concept definition in Mathematics with particular reference to limits and continuity", *Educational Studies in Mathematics*, vol 12, 151 - 169.

Trouche, L, 1992, *Calculatrices graphiques, statut pour l'élève, statut pour le maître*, Mémoire de DEA, Université Montpellier II.

CHILDREN'S MISCONCEPTIONS ABOUT THE INDEPENDENCE OF RANDOM GENERATORS

John M. Truran

University of Adelaide, Australia

The most common erroneous heuristic reported for predicting outcomes from two-outcome random generators is that of negative recency. This paper uses a theoretical model to identify other heuristics, such as the heuristic of positive recency, and heuristics based either on previous predictions or on the success or failure of previous predictions. It reports research which shows that many of these are used by children, and that the heuristic used may also reflect the asymmetry of the random generator.

A Summary of Factors Known to Influence Understanding of Independence

Most research on independence has asked subjects to predict outcomes, usually from a generator known to produce only two outcomes. Children's responses tend to be context specific. They depend on, *inter alia*, age (Turner, 1979; Green, 1982), the level of concreteness of the random generator (Zaleska, 1974), the order in which the random generators are operated (Zaleska, 1974; Fischbein, Nello & Marino, 1991), the way in which the questions are posed (Jones, 1974, pp. 280 - 283) and the amount of information which children have about the random generator (Turner, 1979).

Reported research has found that subjects tend to give responses which are influenced by previous outcomes (Cohen, 1979). Responses may alternate (Green, 1982), may be influenced by some form of negative recency heuristic (Fischbein, 1975, p. 59) or may be using a logical, but overloaded, inductive Baconian approach (Brainerd, 1981, p. 500). But it is also known that different circumstances may influence the popularity of the negative recency heuristic (Turner, 1979) and that the constant predictions required by a Pascalian strategy are superficially so much at odds with the randomness of the actual sequence that many subjects become uncomfortable with such an approach and move to predicting a "representative sequence" (Tversky & Kahneman, 1974) to reduce stress (Goodnow, 1955; Zaleska, 1974). Some students may presume that a representative sequence is what is in fact required (Zaleska & Askévis-Leherpeux, 1976).

In this paper I describe heuristics used by non-Pascalian thinkers for making predictions which are not based on previous outcomes.

Methodology

Interviews were conducted with 32 subjects, four males and four females from each of Years 4, 6, 8, 10 in South Australian schools. Details of the methodology are described in Truran (1994). After two preliminary 'drunken walk' games using a translucent container holding two balls of one colour and one of another, subjects were asked to predict the results of nine successive random draws from this container. No rewards were offered. The colours were chosen to minimise any influence of children's colour preferences; the results here have been standardised to an urn containing two green balls (G) and one blue (B). (The *aide-memoire* 'G for Good and B for Bad', may help the reader to recall that G is the more likely colour.) For variety the questions asked before each draw varied, but all were variations of:

- Which colour is most [sic] likely to come out?
- If I asked you which colour you will draw out which would be the best [sic] colour to guess?
- Which colour do you expect to come out this time?
- Which colour do you think you will take out this time? Why?

There was no evidence that children saw these questions as differing significantly in meaning. Nine draws sometimes became a little boring, especially with a long run of Gs, so this predictive questioning was sometimes omitted, but all subjects wrote down their prediction before every draw.

All draws were made by the child, who also wrote down the result next to each prediction and was then usually asked to comment the outcome using questions like:

- Were you surprised?
- Does that mean you were wrong?
- Why did this happen?
- Can you make it come out blue?
- Did you expect this to happen?
- Will you be right every time?

Again, questions were not asked on every occasion. For example, after three successful predictions of G there was little point in asking for further comment on a fourth success. Outcomes could not be pre-determined, so the protocol had to be flexible.

The preliminary games played before this protocol was administered provided each subject with different experiences, some of which were counter-intuitive. The outcomes of the nine draws were also usually different. Such inevitable variations and the small number of interviews conducted mean that formal statistical analysis of the results is inappropriate. However, the sample is large enough to suggest trends warranting further investigation. All experiences and responses of the subjects have been tabulated in detail. There is only space here for results to be summarised. The analysis below is unashamedly intuitive, but it rests on a process which allows unusual experiences to be readily identified.

Several aspects of the analysis require comment. Identifying heuristics is a statistical process. An heuristic might be said to exist if the probability that Situation A was followed by Response X is greater than some value pre-determined on subjective grounds. The existence of an heuristic does not imply a causal connection between a Situation and a Response, nor does it imply that the subject is conscious of the relation. The work reported here is based on data collected from children with a very wide age-range. Such lumping might not be seen as appropriate in a more refined experiment. But I would argue that it is appropriate in this preliminary study because the whole experiment showed that maturity of thought was not simply related to age.

It was anticipated that subjects thinking in a Pascalian way would predict G consistently and could be presumed to share the received subjective view that the random trials were independent. It was anticipated that the responses of Baconian thinkers would indicate what factors they saw as contributing to the dependence of the trials.

Children's Initial Predictions

All subjects, except two in Year 4, predicted that the first ball drawn would be G. TD (Year 4, M, 9:3) expressed several times a very strong liking for B so it is likely that affective matters influenced his choice. While he was aware of the asymmetry he saw the presence of 2G as "cheating". On the other hand, MW (Year 4, F, 8:7) did not seem to be aware that the asymmetric composition of the urn was relevant. She made no comment on the relative numbers in the urn, but made comments like "Because it looks like it's the green's turn today" and "It looks like it's green before blue".

These results provide some support for Turner's (1979) suggestion that it is not until a child is about 9 years old that he or she appreciates the significance of such asymmetry. It will be argued below that once this appreciation has developed the heuristics adopted are likely to be influenced by this asymmetry.

Existence of Pascalian Responses

Surprisingly, five subjects, one in Year 4, one in Year 6, two in Year 8 and one in Year 10, consistently predicted G for all 9 draws even though all of these had at least two of their predictions refuted during the experiment. Five of the predictions of ML (Year 4, M, 8:10) were unsuccessful. When asked why he had chosen G he replied, "Same as the other questions, cos there's 2 Gs." When asked whether he could make the result a G he replied in the negative "because it's hard to choose cos I know where it is at first, but when you shake it up it's hard to find them".

The other subjects gave similar, though more mature responses; all showed a clear awareness that G was the more likely colour, but that B would sometimes occur. LH (Year 8, M, 13:1) explained the occurrence of a B by saying, "Just luck of the draw. You've got 33% chance of a B to come out or a G to come out because you've got double G and one B and you've got more chance of getting a G but you can get a B."

So, under the conditions of this experiment, Pascalian responses indicating an understanding of the independence of random trials may occur as early as Year 4, but are not common even in Year 10. We turn now to children's non-Pascalian heuristics.

Were the Subjects Using an Heuristic of Alternating Responses?

Examination of all the transcripts shows that the answer to this question is unequivocally "No". As mentioned above, almost all subjects were aware of the asymmetry of the random generator, and showed some understanding that this affected each outcome. But even the two exceptions, MW and TD, did not use an alternating response heuristic. As far as I can tell researchers reporting the use of this heuristic have been working with two-outcome random generators which are either symmetric or whose rule is unknown to the subject.

What Factors Could Influence Subjects' Predictions?

In this protocol, after five draws from the urn, each subject would have written in front of him or her a set of results laid out as in the example in Table 1. In making the sixth prediction the subject could make use of many different components of the table. Some of these possibilities will be enumerated, and then used to show that the heuristics used by subjects are more diverse than might be apparent at first sight.

Prediction	Actual
G	B
G	G
B	G
G	G
G	G

Table 1

The sixth prediction might be statistically based on some or all of the data in the 'Actual' column. This would imply the use of some form of 'recency heuristic'. A 'negative recency heuristic' exists when "the probability of predicting an event [decreases] as a consequence of the event having occurred repeatedly on previous trials" (Fischbein, 1975, p. 59). For example, after 6G the subject is less likely to predict a B than after just 1G. Fischbein's term 'consequence' implies causality; a preferable definition would be "the probability of predicting an event decreases as the number of previous occurrences of the event increases".

Equally, the sixth prediction might be statistically based on some or all of the data in the 'Prediction' column. This is like a recency heuristic, but follows predictions rather than outcomes, so is necessary to define a new term: 'recency heuristic for predictions'. Alternatively, the sixth prediction might be based on some or all of the confirmations or refutations of the previous predictions. In the example given, the second, third and fourth predictions were confirmed, and the other two refuted. Finally, given that B and G are not equally likely, the heuristics might be influenced by the asymmetry of the outcomes. If every entry in the table were replaced by its opposite colour the heuristic adopted might or might not be the same.

Were the Subjects Using an Heuristic of Negative Recency ?

Some subjects certainly used a negative recency heuristic. For example, AP (Year 8, F, 13:8) changed from predicting G to predicting B after four of her six G predictions had been refuted by a B outcome because she "[seemed] to be getting B all the time".

So the data may be examined to look for relationships between the length of run and the predictions made. In this paper "run" is defined so that the sequence GGBGGG contains runs of 1B, 2G and 3G). Other definitions are possible, e.g., AP's reasoning quoted above, but this one has been chosen because it is simple and therefore appropriate to a preliminary investigation. In the analysis which follows the five subjects who were probably using a Pascalian strategy are not included.

All subjects' predictions after various lengths of runs of B and G have been tabulated. The percentages of no change in prediction, changes from G to B, and changes from B to G for after each run have been calculated. Because runs are cumulative (i.e., a run of three contains runs of two and one) there are far more examples of short runs than of long ones. Cases with less than 10 examples are excluded from the analysis unless specifically stated otherwise.

The probability that a subject will make a change of prediction from G to B is about 22% (range 18 - 25) for runs of between 1G and 4G. Similarly, the probability that a subject will make a change of prediction from B to G is about 25% (range 22 - 29) for runs of between 1G and 4G. Finally , the probability that a subject will make no change of prediction is about 53% (range 47 - 60) for runs of between 1G and 4G.

But of the 27 runs of 1B, 33% were followed by a change in prediction from G to B, 33% from B to G, and 33% by no change. Of the 9 runs of 2B, 67% were followed by a change in prediction from G to B, 22% from B to G, and 11% by no change.

It can be seen that after runs of G there is no evidence of the use of a negative recency heuristic. Subjects are just as likely to change from G to B or from B to G after a moderately long run of Gs as after a short run. But for runs of B there is some evidence that a positive recency heuristic may be being used for changes from G to B.

So these figures do not support the thesis that a negative recency heuristic is used for prediction in these circumstances, and suggest that sometimes a positive recency heuristic may be employed. The data suggest that the choice of heuristic may be influenced by the asymmetry of the colours so this possibility will be examined next.

Does Asymmetry Influence the Heuristics Employed?

There may be a tendency for subjects to leap on a bandwagon when an unexpected sequence of results occurs. After short and moderately long runs of G the difference in the relative frequency of changes from B to G and from G to B is very small. The difference in changes after a run of 1 B is very small, but after a run of 2B 67% of the predictions were a change from G to B, while only 22% were from B to G. In spite of the small numbers involved, the difference is striking.

As well as comparing the relative frequency of the direction of changes after different runs of a given colour, it is also possible to compare changes after the same length of run of different colours. Occurrence of a single B was 33% likely to be followed by a change from G to B, whereas occurrence of a single G was only 23% likely to be followed by a change from B to G. Even more strongly, a run of 2 Bs was 67% likely to be followed by a change from G to B, whereas a run of 2 Gs was only 18% likely to be followed by a change from G to B. This difference seems important even though there were only 9 runs of 2B. So for some subjects occurrence of the less likely outcome is more likely to be followed by a change in prediction towards the less likely outcome than is the case for occurrences of the more likely outcome.

Further evidence of colour asymmetry influencing heuristics may be found by comparing the probabilities of there being no change in prediction after various runs of G with those after runs of B. The lowest figure for no change after a run of Gs is 47%, and the highest figure for no change after a run of Bs is 33%.

So there is strong evidence that colour asymmetry influences subjects' heuristics. But other factors also may be at work. For example, it is possible that subjects may be using a negative recency heuristic based, not on the actual outcomes of the trials, but on their predictions.

Does the Negative Recency Heuristic Operate with the Predictions?

Subjects predicted 116 runs of G. Of the runs of length 1 or 2, 33% were followed by a change to a prediction of B. Of the runs of length 3 or 4, 45% were followed by change, and 50% of longer runs were followed by change. Of the 51 predicted runs of B, 80% of runs of 1 were followed by change and 88% of the 9 cases of runs of 2.

The percentage differences here are small, but, in terms of changing from G to B, there is more evidence that students are applying the negative recency heuristic to their predictions, than to the outcomes. In some cases this assertion can be supported by individuals' comments. JM (Year 8, F, 13:7) changed after 6 predictions of G of which the third was refuted by a B outcome "because it's been G all the time and its due for a change". Similarly, JW (Year 10, F, 15:3) changed after eight predictions of G of which the first two were refuted because "I only picked B because it's been all Gs before". Predictions and outcomes were written down by the subjects and accessible to them, so there is strong evidence that previous predictions were sometimes used as a basis for a next prediction. and perhaps using a negative recency heuristic.

But it is also possible that subjects' predictions were related to the conformity of their predictions with actual outcomes.

Were the Subjects Using an Heuristic Related to Refutations?

Goodnow (1958, p. 115) observed that "[f]or most individuals, correct prediction of an infrequent event is not just one more correct prediction but a success worth several correct predictions of the easy-to-get more frequent alternative". We consider first whether just one refutation of a prediction has any influence on the next prediction.

A refuted prediction of G was 25% likely to be followed by a change of prediction to B. But equally a confirmed prediction of G was 20% likely to be followed by a change of prediction to B. On the other hand, a refuted prediction of B was 71% likely to be followed by a change of prediction to G whereas a confirmed prediction of B was 75% likely to be followed by a change of prediction to G. The figure of 71% is deflated in this sample by TD's animistic fixation on B mentioned above.

Colour asymmetry remains important here, with a prediction of B being about three times as likely to be followed by a changed prediction as is a prediction of G. But refutation or confirmation of a single prediction of either B or G is a poor predictor of whether the subject would change his or her prediction for the next draw. But when several refutations are considered the situation changes strikingly.

There were 39 changes of prediction from B to G. Of these 9 (23%) occurred after a confirmed prediction of B, 11 (28%) after exactly one refuted prediction of B and 19 (49%) after 2 or more refuted predictions of B. Similarly, there were 39 changes of prediction from G to B. Of these 25 (64%) occurred after a confirmed prediction of G, 8 (21%) after exactly one refuted prediction of G and 6 (15%) after 2 or more refuted predictions of G.

These differences are striking and support Goodnow's assertion. Confirmed prediction of G predicts a change to B much better than confirmed prediction of B predicts a change to G. Refutation of two B predictions predicts of change much better than refutation of two G predictions.

Discussion

The protocol used has weaknesses, but it does bring out some points clearly. It allowed almost all children to respond in ways which showed that they were well aware of the asymmetry of the random generator.

It also showed that a negative recency heuristic is less commonly used than has been suggested and revealed other possible heuristics. Predictions do not seem to be affected by G outcomes, but there may have be a positive recency heuristic employed with respect to B outcomes. A negative recency heuristic may be used with respect to predictions of G, rather than outcomes. While a single refutation of any kind was a poor predictor of change for the next prediction, two refutations of a B prediction were a very good predictor of a change to a G prediction. Surprisingly, a confirmed prediction of G was also a good predictor of change.

Some of the heuristics observed here have not been observed by other workers. Perhaps this is because protocol like the one used here are rarely used. They are labour intensive, unable to ensure constant experiences for all subjects, and cannot easily be extended to large numbers of draws.

The major weakness of the protocol is that it does not make quite clear whether the subject is asked to make predictions which will match the actual outcome, or try to predict a "typical" outcome, or try to maximise rewards under pay-off conditions which are not explicitly stated. There is evidence that many subjects were trying to do all three. JW's decision, quoted above, to change in order to include one B in her sequence is strong evidence that she felt that she was required to produce a "typical" sequence. The protocol would be of greater benefit if it made clear to the subjects that a strategy which maximises rewards is what is required. Teigen (1983) has shown that such a modification can be easily implemented.

Independence of random generators is an important statistical idea which it is desirable for children to develop. This paper has shown that they develop many erroneous intuitions, and points to issues which need to be addressed in classroom teaching.

References

Brainerd, Charles J. (1981) "Working Memory and the Developmental Analysis of Probability Judgement" *Psychological Review* 88(6): 463 - 502

Cohen, L. Jonathan (1979) "On the Psychology of Prediction: Whose is the Fallacy?" *Cognition* 7: 385 - 407

Fischbein, E. (1975) *The Intuitive Sources of Probabilistic Thinking in Children* Dordrecht, Holland: D. Reidel

Fischbein, Efraim, Nello, Maria Sainati & Marino, Maria Sciolis (1991) "Factors Affecting Probabilistic Judgements in Children and Adolescents" *Educational Studies in Mathematics* 22: 523 - 549

Goodnow, J.J. (1955) "Determinants of Choice-Distributions in Two-Choice Situations" *American Journal of Psychology* 68: 106 - 116

Goodnow, Jacqueline Jarrett (1958) "A Review of Studies on Probable Events" *Australian Journal of Psychology* 10: 111 - 125

Green, David R. (1982) A Survey of Probability Concepts in 3000 Pupils aged 11 - 16 Years in Grey, D.R., P. Holmes, V. Barnett, G.M. Constable (eds), *Proceedings of the First International Conference on Teaching Statistics* Sheffield, U.K.: Teaching Statistics Trust

Jones, Graham A. (1974) *The Performance of First, Second and Third Grade Children on Five Concepts of Probability and the Effects of Grade, I.Q. and Embodiment on their Performance* Ph.D. Thesis Indiana University

Teigen, K.H. (1983) “Studies in Subjective Probability I: Prediction of Random Events *Scandinavian Journal of Psychology* 24: 13 - 25

Truran, John M. (1994) "Examination of a Relationship between Children's Estimation of Probabilities and Their Understanding of Proportion" pp. 337 - 344 in Vol. IV of da Ponte, J.P & Matos, J.F (eds) (1994) *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education* Lisbon, Portugal: Program Committee of the 18th PME Conference

Turner, Irené F. (1979) "Event Probabilities and Information as Determinants of Children's Sequential Behaviour" *Irish Journal of Psychology* IV (2): 95 -100

Tversky, Amos & Kahneman, Daniel (1974) "Judgement under Uncertainty: Heuristics and Biases" *Science* 185: 1124 - 1131.

Zaleska, Maryla (1974) "Influence du mode de présentation de séries aléatoires sur le choix d'une stratégie: I.—Situations hypothétiques" *L'Année psycholigique* 74: 125 - 144

Zaleska, Maryla & Askévis-Leherpeux, Françoise (1976) "Influence du mode de présentation de séries aléatoires sur le choix d'une stratégie: II.—Situations réelles" *L'Année psycholigique* 76: 501 - 514

CHILDREN'S USE OF A REPRESENTATIVENESS HEURISTIC

Kathleen M Truran

University of Adelaide, Australia

This paper describes an investigation by Teigen into the use of a representativeness heuristic by adults and a replication with young children of Teigen's investigation. A comparison of the data from both studies indicates that in two cases a similar heuristic appears to be used by both adults and children, in a third the responses are very different. This raises a question about which heuristics and biases children may use and how these differ from those of adults.

The Representativeness Heuristic

The thesis of Tversky & Kahneman (1982) is that people who are statistically naïve make estimates for the likelihood of events by using a heuristic which they called representativeness. An example is the belief that if a coin is tossed and comes down heads eight times in a row the next toss must produce a tail. This belief implies that such a result would produce a pattern seen as being more representative than nine heads. Occurrences of permutations and combinations are also often interpreted by the use of a representativeness heuristic, with the sequence HTHTTH viewed as more likely than either HHHTTT or HHHHTH. While sometimes this heuristic works well, an exclusive reliance on it often leads to inappropriate decisions made in mathematics and daily life. It is important that such misconceptions are replaced by more appropriate thought processes.

Focus of Study

Teigen (1983) carried out an investigation involving tertiary students who answered three questions which he devised to test his hypothesis that a representativeness heuristic is used when people are asked to predict the outcome of a single, random event. As a result Teigen claimed that when answering such questions most people make their answers *look* spontaneous as well as being representative of the population. Teigen's claim is related to anecdotal evidence which suggests that when people are asked to choose a number between one and ten, five is avoided because it is too obviously the middle number, and the most frequent answer given is seven.

Teigen's three questions seemed to be easily understood, to investigate different types of responses and heuristics, and appropriate for use in both group and individual situations, so I decided to investigate the responses of primary school children to these same questions.

Teigen's Research

At the beginning of a scheduled lecture, the tertiary students were asked the following three questions:

> Q1 These twelve tickets numbered one to twelve are folded and put into this container. There is only one ticket for each number, so all tickets have an equal chance to be drawn. Still, make a guess and write down which number you think will be drawn.

Teigen claimed that these students knew that all outcomes were equally probable and understood elements of probability; even so, the majority of students selected the central values (5, 6, 7, and 8). Teigen claimed that this indicated the use of a representativeness heuristic.

> Q2 These six red tickets numbered one to six and these blue tickets numbered seven to twelve are folded and put into this container. There is only one ticket for each number, so all tickets have an equal chance to be drawn. Still, make a guess and write down which number you think will be drawn.

When the tickets were divided into two groups Teigen observed a different pattern of responses. 'The central tendency had disappeared ...[,] subjects seemed rather to prefer one of the middle blue numbers (3 or 5), or a representative red one (9)'.

> Q3 These twelve tickets numbered one to twelve are folded and put into this container. There is only one ticket for each number, so all tickets have an equal chance to be drawn. Still, make a guess and write down which number you think will be the last number to be drawn [from the container] after the first eleven have been drawn one by one.

Teigen observed that 'To guess the last number drawn from a box seems to be rather different from guessing the first one. The distribution of guesses is by now almost rectangular, with extreme values quite as frequently chosen as more central ones.'

Teigen's sequence of experiments ...'were designed as a first attempt to clarify whether a representativeness heuristic applies when subjects are asked to predict simple, equally probable random events.' He used a 'show of hands' to indicate the choices made. This method is not the most effective for collecting a set of data of this size, and Teigen himself suggested that 'some answers, especially the most frequent alternatives might have been lost.' He also questioned whether the same students participating in more than one question might have influenced their answers but added, '[S]till the results strongly suggest that superficial structural and procedural differences will lead to different patterns of guesses.'

Methodology

The questions used by Teigen were replicated in this study, which involved about 280 students in Years 5 and 7, and 28 in Year 3, all from six co-educational schools in lower, middle and upper socio-economic regions.

The testing procedure differed from that of Teigen. Years 5 and 7 students were questioned in class-sized groups of about 30, and wrote their responses onto formal recording sheets. A random selection of six from each class was later interviewed individually, as were all Year 3 students. While each question was being asked raffle-tickets and urns as appropriate were demonstrated to the students.

In the first class an attempt to use Teigen's questions*verbatim* caused difficulties of interpretation by some children, who interrupted and asked for clarification. I decided that these difficulties required an explanation which clarified that students were expected to 'choose a number'. Having carried out this explanation with one class I decided to use the same explanation before asking the questions in all later situations.

Use of recording sheets allowed me to identify students who made the same choice for all three questions and/or wrote reasons for their choices. Use of interviews allowed me to probe reasons for the choices made.

Analysis of Responses

The responses to question 1 which are shown in Table 1 enable a comparison to be made between those made by the children involved in this study and those made by the adults in Teigen's study. The results are shown as percentages; there may be some small errors due to rounding. The central values (5, 6, 7, 8) are printed in bold to identify the numbers defined as those indicating use of a representative heuristic as proposed by Teigen.

Teigen has not reported any incidence of an 'alternative choice' in his study. It has been included as part of this study because it was made so frequently by Year 5 and 7 students in this and other questions. The formal recording sheets made it easier for those with alternative views to express them, even though they were not encouraged.

For this question Teigen's view is supported by the responses of students in all three year groups. However, the choices of extreme values made by

Year 5 students seem to vary from those of all other groups, by being biased to the upper numbers (9, 10, 11, 12).

Table 1

Prediction about the result of the first ticket drawn from tickets numbered 1-12, given as percentages

	Truran			**Teigen**
Numbers	Year 3 n = 28	Year 5 n = 167	Year 7 n = 115	Adult n =201
1,2,3,4	18.7	9.4	21.2	19.2
5,6,7,8	**52.3**	**52.1**	**50.1**	**58.9**
9,10,11,12	28.3	31.2	18.6	21.9
alternative choice		5.2	10.4	

The primary students' responses to Question 2 (Table 2), where the twelve numbers are split into blue and red groups, form a very different pattern from the adults' responses; where Teigen argued that the central tendency had disappeared, and that the preferences were divided between the two groups with peaks appearing at two 'blue' and one red 'representative' number.

In all three groups of primary students, there is still evidence of a distinct preference for the central values (5, 6, 7, 8) as for Question 1. Of those Year 3 and 5 students who did not choose central values there is a bias towards the upper values (9, 10, 11, 12). However, this is not the case with Year 7 students whose choices of both sets of extreme values are symmetric. The 'alternative choice' is particularly large in the case of Year 7 students and makes the gauging of an accurate indication of their beliefs difficult.

Table 2

Prediction about the result of blue tickets numbered 1-6 and red tickets numbered 7-12, given as percentages

	Truran			**Teigen**
Numbers	Year 3 n = 28	Year 5 n = 167	Year 7 n = 115	Adult n = 201
1,2,3,4	20.7	20.8	19.1	26.5
5,6,7,8	**43.9**	**42.2**	**46.4**	**36.5**
9,10,11,12	36.8	33.4	19.1	35.0
alternative choice	1.0	5.7	12.9	

Question 3 (Table 3) produced very different responses from the previous questions. Because of the different patterns of responses to Question 3 (Table 3), the lower values (1, 2, 3, 4) and (12) are printed in bold to enable efficient comparison of the numbers which were most frequently chosen by the primary students.

When the adult students were asked 'what number do you think will be drawn after the first eleven tickets have been drawn one by one', the pattern of choosing central values diminished and the choice of extreme values was almost as frequent.

The primary students' responses, on the other hand, indicated a strong preference for the lower values (1, 2, 3, 4) and for 12. From their written and interview comments these choices seemed to be based on a belief that there is a positional relationship between the number on a ticket and its place in a container.

Possibly the only pattern that emerges from comparison of the data from this question is the decline in the choice of (12) after Year 3 in this study. Again, the large 'alternative choice' selection by Year 7 students has had a marked influence on the overall results.

Table 3

Prediction about the result of the last ticket drawn from tickets numbered 1-12, given as percentages

	Truran			**Teigen**
Numbers	Year 3 n = 28	Year 5 n = 167	Year 7 n = 115	Adult n = 201
1,2,3,4	**39.1**	**29.8**	**45.8**	**30.0**
5,6,7,8	19.2	25.4	25.6	34.0
9,10,11	13.1	20.1	9.4	27.0
12	**25.0**	**19.1**	**10.5**	**7.0**
alternative choice		2.9	10.3	

Analysis of Children's Detailed Responses

Once the written responses from students taking part in this study were analysed it became evident that some primary students, particularly those in Year 7, were reluctant to choose one number to predict an answer to one or all of the three questions, and chose instead a non-numeric response. Interviews gave me an opportunity to examine responses to questions in more depth and to probe the responses given.

ALTERNATIVE CHOICES

Alternative choices were made by as many as 6% of Year 5 and 13% of Year 7 students. 'I don't know' was a common answer. When the student had made the choice 'any', it was qualified by responses like, 'you can't tell which number you will pick, they all have the same chance'. A number of students discussed the process of mixing the tickets to clarify the point being made.

(LA F. Yr 7 12:2)

LA It could be any number.
I Can you say why?
LA No all numbers have the same chance if they're mixed up.

The most frequent explanation was a criticism of the wording of the questions: they 'didn't make sense', therefore, you couldn't say what number 'it' would be.

The difficulty for some primary children seemed to stem from the hypothetical problem presented by the questions and the difficulty of conceptualising the situation. But there are others who had no trouble with this, and made a specific number choice and confidently justified their choices.

THE LARGEST, SMALLEST AND MIDDLE NUMBERS

Common specific number choices were six, twelve or one. The reasons given for these choices related either to the fact that these were the largest, smallest or middle numbers of the set, and thought that the number on this ticket would put it into an optimal position in the container, so that it would be the first (or last) chosen.

The following interviews were with students who chose what they saw as the middle number and explained why they had made this choice.

(DR F. Y7 11:10)

DR Six.
I Can you say why?
DR When it's something like this I always choose the middle number.

(CL F. Y5 10:2)

CL Six.
I Can you say why?
CL Because it's halfway.

The students who chose extreme values, appeared to be using an image of the arrangement of tickets in a container, and so, in trying to make sense of the questions they focused on a mental picture to help them to do this.

(TC M. Yr 5 11:0) replied

TC Twelve.
I Any reason for choosing 12?
TC It's the highest, most of the time you take out the highest last.

(KP. F. Year 3 8:9)

KP Twelve.
I Can you say why?
KP Because there are 12 tickets altogether, so I think 12 would be the last number left.

(AN M. Year 5 11:2)

AN Twelve.
I Can you say why?
AN Because it would be on top it would be the last to go in.

DECISIONS REFLECTING THE ENVIRONMENT

Some children responded to the questions as if they were raffles. Again personal preferences, or past experiences influenced a number of choices which the children could neither explain or justify, except by implying that the number represented an important one.

(MA F. Y7 11:3)

MA Ten
I Why would ten be your choice?
MA Because ten or any other higher number with two numbers together, like eleven or twelve, the highest numbers just come up in raffles and the lower numbers don't and when the numbers are all shuffled up and all that the higher numbers are mostly on top.

(AD M. Y.5 9:1)

AD I'd choose 3 or 4; sometimes when there's a raffle I often get 4. Once there were two raffles one for Joey Scouts and one for Cubs and I picked out 4 two times in a row.

(MM M. Y3 7:10)

MM Six.
I Can you say why?
MM It's my favourite number.

(JI F. Y5 10:3)

JI Ten
I Can you say why
JI It's how old I am

Discussion

These questions caused a great deal of difficulty for some students. Some complained 'I've never done this before' and were reluctant to respond. The use of two colours in question 2 seemed to influence some responses, especially from younger children, who interpreted the use of two colours as relating to two separate questions. They asked questions like 'can I do 2 colours' (select a 'blue' number and a 'red' number)?

Teigen's claim, supported by his research, is that a representativeness heuristic determines the middle-range choices made by the majority of subjects questioned. While there is evidence of some middle-range choice by young children there is also evidence of other influences on their responses to questions posed in this study.

A comparison of responses to Questions 1 and 3 would provide a basis for an interesting further study. An analysis of the very different responses to these questions will provide worthwhile information on how children perceive two questions that are similar; which ticket will be drawn first, which will be drawn last? While at the same time providing some further insights into how children view non-determanistic situations of this type.

References

Teigen, K.H. (1983) 'Studies in Subjective Probability I: Prediction of Random Events' *Scandinavian Journal of Psychology* 24: 13-25.

Tversky, Amos & Kahneman, Daniel (1974) 'Judgement under Uncertainty: Heuristics and Biases' *Science* 185: 1124 - 1131.

I acknowledge the generous support given by the Research Management Committee of the Faculty of Arts, University of Adelaide, in enabling me to participate in this conference.

TWO PROBLEMS UNDER ONE TITLE:
THE CASE OF DIVISION BY ZERO

Pessia Tsamir

Kibbutzim State College and Tel Aviv University

*Division by zero, is often regarded as problematic for instruction. So far little has been done to find out whether students distinguish between **a:0 for a≠0 and 0:0**, and whether they know the different reasons for each of those divisions' being undefined. This article reports an analysis of 32 preservice (secondary) mathematics teachers' understanding of division by zero. The findings show that, although many of the participants could produce correct answers to the problem 5:0, several could not, and only a few were able to give mathematical explanations for the underlying principles. Moreover, even though the case of a:0 for a≠0 was discussed in class through several didactic approaches, when asked to solve the problem 0:0, the participants encountered difficulties. However, quite a number of participants concluded that 0:0 is undefined, while none of them could sensibly apply the formal explanation.*

Introduction

The fact that division by zero is undefined has been indicated by researchers as problematic for instruction (see, for instance, Allinger, 1980; Ball, 1990; Reys, 1974; Wheeler & Feghali, 1983). Some of the reasons for this, as research indicates, might be rooted in mathematics education. Young students, when dealing with division, are confronted for the first time with a mathematical situation that is excluded as "illegal": the case of division by zero. Until then students' mathematical experience has guided them to believe that every problem has a numerical solution, so that an unsolved problem must be the consequence of their poor comprehension and their being unable to provide the probably existing desired solution.

Findings indicate students' tendency to attribute numerical values, finite or infinite, to a:0 (for both a≠0 and a=0). Most students who held that a:0 was undefined based their answers on memorized rules only (e.g. Ball, 1990; Reys, 1974).

Students' problem with division by zero is two-fold as this division in fact includes two different parts: **a:0 for a≠0** and **0:0.** This is obtained by carefully studying the definition of division. The quotient a:b is the *unique* number x such that b·x=a, *if such a number exists*. An attempt to apply this definition with b=0, for either a≠0 or a=0, yields the following conclusions: (1) Let a≠0. Then a:0 means a number x such that 0·x=a. But , clearly no such number can exist since 0·x always equals zero, whereas a≠0 is given. (2) Let a=0. Then a:0 or 0:0 means a number x so that 0·x=0; But this equation is satisfied by any number x, and hence the problem *does not have a unique result.* Thus, division by zero is excluded in all cases.

As mentioned before, these two different explanations account for the fact that the first encounter students have with 'the undefined' is 'double trouble': Within one mathematical context and under one title, two kinds of undefined cases are concurrently presented, in a visually similar way. Not only are the students required to and accept the meaning of 'undefined' in mathematics; but they must also distinguish between the factors that have caused these two different cases to be undefined.

To the best of our knowledge, there is no reported evidence about either students' or preservice teachers' understanding of the mathematical meaning and the special problematics of 0:0. Nor have we found evidence about their ability to distinctly infer to attributes that differentiate 0:0 from a:0 in the case of a≠0.

This lack of evidence determines the aims of this study, which are (a) the investigation of preservice teachers' ideas concerning 5:0 and 0:0, and (b) the connections, if any, that have been made in the case of 0:0 with the formal explanation that was presented in class for 5:0, as a representative of a:0 for a≠0, being undefined.

Method

A group of 32 preservice (secondary) mathematics teachers were selected for:

(a) ***An individual interview*** - The students were asked to solve 5:0=? and 0:0=? in writing, in the presence of the researcher, and then to discuss their intuitive answers in an individual interview. (b) ***A short course concerning the problem a:0 a≠0*** - Students participated in a course consisting of three main parts using: (1) the formal approach, (2) some additional approaches, as introduced by Knifong & Burton (1980), and (3) guiding towards analyzing each given answer by focusing on problematic ideas and misconceptions. (c) ***A questionnaire*** - About five weeks after the course, the participants were asked again to respond, in writing, to a questionnaire that included the problems 5:0=? and 0:0=?

It is noteworthy that the problem of 0:0 was not referred to at all during the course.

Results

The results are displayed in two main columns: pre-course and post-course responses (Table 1). For each column judgments and justifications to the problems 5:0 and 0:0 are presented, and new ideas or special expressions that were raised are detailed.

Table 1

Judgments and justifications to 5:0 and 0:0 - before and after the course

	Pre-Course		Post-Course	
	5:0	0:0	5:0	0:0
Undefined	**21**	**19**	**32**	**25**
Valid formal explanation	3	-	25	-
Invalid explanation	-	-	-	3
"It's a rule" (memory)	8	9	3	15
"Zero is nothing"	2	2	-	-
Everyday considerations	5	4	2	4
Locally/temporarily undefined	3	4	2	3
Zero	**2**	**5**	**-**	**3**
Overgeneralization 0:a, a≠0	2	3	-	2
Everyday considerations	-	2	-	1
Five	**2**	**-**	**-**	**-**
"Zero is nothing"	1	-	-	-
Everyday considerations	1	-	-	-
One	**-**	**3**	**-**	**1**
Overgeneralization 0:a, a≠0	-	2	-	1
Everyday considerations	-	1	-	-
Infinity	**5**	**3**	**-**	**2**
"It is the limit"	5	3	-	2
Don't Know	**2**	**2**	**-**	**1**

I. Pre-Course Responses

Preservice teachers' answers to the problem 5:0

Twenty-one of the 32 participants offered correct judgments, arguing that **dividing five by zero is impossible** or **undefined**, but their explanations varied.

Only three of these students used the ***formal justification*** for their answers (see a1 below). In most cases the correct responses were not accompanied by meaningful justifications (a2-a5). For instance, participants tended to base their performance on particular rules rather than focusing on underlying meanings; thus eight of them based their answers on ***sheer memorization*** with no meaningful explanation (a2). Seven participants exhibited an ***erroneous conception of zero being 'nothing'***, rather than a numerical entity. Two of them justified their claim purely within the mathematical framework (a3), whereas the other five inappropriately added ***realistic (or everyday) examples*** (a4). A rather interesting line of reasoning was displayed oy three participants who viewed division by zero as being ***locally undefined*** in a limited mathematical field or ***temporarily undefined***, as long as no calculus has been studied (a5).

(a1) 5:0=? has 0·?=5 as a related multiplication sentence... Therefore, 5:0 has no solution.

(a2) My teacher explicitly emphasized that there is such a rule in mathematics...

(a3) It is impossible to divide by zero, as you have nothing to divide by, hence there is no result to 5:0.

(a4) You cannot perform the division of five biscuits, in equal shares, among no children.

(a5) Until a certain age you must use the word 'undefined'. It's just that you can't tell young children at school about infinity. They won't understand it.

In addition to the correct responses, various ERRONEOUS RESPONSES, were presented by nine participants who attributed numerical values to 5:0. Five of them made ***inappropriate connections with the theory of potential infinity***, concluding that ***5:0=infinity*** (b4). They confused the 'static' procedure of division, with the 'dynamic' procedure of limits. It is noteworthy that the issue of limits had been studied during this period in calculus lessons.

Two participants overgeneralized the relation between multiplication and division (b1). Their response was that dividing and multiplying anything by zero equals zero, thus **5:0=0**. Another couple of students claimed that 5:0=5 (b2,b3) due to their ***misconception of zero*** (assuming it was 'nothing'), and their ***misconception of division*** (confusing the remainder with the quotient).

(b1) 5:0=0 - As the result of multiplying anything by zero is constantly zero, dividing anything by zero gives the same result.

(b2) 5:0=5 - 5:0 means 5 divided by nothing. I haven't divided by anything, so I still have 5.

(b3) 5:0=5 - Dividing 5 marbles among nobody leaves you with your five marbles.

(b4) 5:0=infinity - The smaller the divisor, the bigger is the result... It's the limit... infinity.

(b5) I cannot figure out which one is problematic... dividing zero or dividing by zero...

Unsuccessful attempts, made by two other participants who relied solely on their memory, produced **'no answers'**. Their explanations revealed their confusion which was caused by having no conceptual understanding to rely on (b5).

Preservice teachers' answers to the problem 0:0

Nineteen of the 32 participants presented CORRECT JUDGMENTS to the problem 0:0, i.e., that **0:0 is undefined**. However, none of their inferences were based on the formal explanation. In nine cases the justifications were merely consisted of stating ***a certain rule***, without any indication that the participants knew what the rule was. Much like in the case of solving 5:0, six participants exhibited ***wrong conceptions of the number 'zero' and the operation 'division'***, and four more participants based their assertions only on ***everyday experience*** (c1).

(c1) If one has no marbles and there are no children to divide the (no) marbles to, then how can you possibly give anything to anyone? It's impossible.

Four participants presented again the idea of ***locally undefined*** (i.e., the answer depended on one's mathematical background, so that 0:0 would equal 'infinity' for calculus graduates and 'undefined' for those who were not acquainted with the concept of limit).

However, not all participants claimed that 0:0 is undefined. Thirteen participants came up with erroneous responses. Three of them argued that 0:0 equals **infinity**, whereas eight presented the following finite solutions **0:0=1** (3), and **0:0=0** (5). These solutions were due to their ***overgeneralization of mathematical rules***, such as ***a:a=1*** and ***0:a=0*** (c2, c4). In one case, a ***realistic (everyday) example*** was provided (c3).

(c2) 0:0=0 - Zero multiplied by any number and zero divided by any number is zero.

(c3) 0:0=0 - If you have, for example, no cake and no children to give them from it, then there is nothing.. no cake and no children.. zero of everything..

(c4) 0:0=1 - Any number divided by itself is 1.

Two participants had **no answer**, as they claimed neither to remember the solution nor being able to reconstruct it.

II. Post-Course Responses

Preservice teachers' answers to the problem 5:0

After the course, all 32 participants claimed correctly that 5:0 is undefined, and 25 of them justified their claim via the ***valid formal explanation***. Still, in their justifications, three participants were satisfied with merely stating that ***there is such a rule***, two other participants only presented an ***everyday illustration***, whereas the last two participants explained that as long as infinity and limits had not been studied 5:0 would be ***"temporarily undefined"***.

It is most striking that even after the course some participants failed to understand that realistic examples are quite problematic. Such examples are not always faithful to the mathematical ideas for which they are meant to stand. Thus, for instance, if we accepted the idea of "dividing marbles" - 0:5 should also be undefined, because dividing no marbles among five children is practically impossible.

Preservice teachers' answers to the problem 0:0

The improvement in preservice teachers' responses to 0:0 after the course was not as dramatic as in the case of 5:0. Still, it was quite meaningful considering the fact that 0:0 had not been explicitly dealt with at all during the course. Twenty-five participants answered correctly that **0:0 is undefined**, but they did not provide any **valid explanations**. Fifteen of them arbitrarily claimed that ***there is such a rule***, four others provided ***realistic examples***, and the remaining six argued that the answer would be ***locally undefined*** so long as we cannot refer to calculus and infinity Though three participants did try to apply the formal process which they had studied during the course, all of them failed to do it right. They submitted the following ***invalid explanations***:

(d1) We have already shown that 3:0=t⇔t·0=3 that's impossible so 3:0 is undefined... Similarly, I would assign the students to complete the procedure 0:0=t⇔t·0=0 so that 0:0 is undefined.

(d2) I'd start with a familiar example such as 2:0=x implies that x·0=2, which contradicts the necessity to get zero and consequently the response should be undefined. Now 0:0=y means that 0·y=y - contradiction!

These participants were quite sure about "knowing the answer", and as they were asked to justify their claim, they hastily fabricated some apparently formal explanation. They started with implementing the procedure that was used in class for a≠0, but consequently two lines of reasoning were introduced by them in the following manner:

a. **Presenting valid connections as contradictory** - It can easily be seen (d1) that no contradiction has actually been reached

b. **Reaching the contradiction through invalid connections between multiplication and division** - A wrong claim that *0:0=y implies 0·y=y*, creates a "negation" (d2).

Six preservice teachers related to 0:0 numeric values, that are **zero, one** and **infinity**. Three participants based their justifications on the overgeneralized rule **0:a** also using **everyday considerations**; one participant justified his claim by the over generalized rule **a:a=1**; and 2 others justified by the **rules of limit**. One participant admitted that he **did not know** the answer (d3):

(d3) I can't decide whether it follows the rule that it's zero, or if it is undefined. There may even be another rule for this special case of two zeros.

Final Comments and Recommendations

Our findings indicate that preservice teachers failed to make a meaningful transfer from the case of a:0 for a≠0, which was studied in class, to 0:0 being undefined. In several cases they even expressed their confusion, being not quite sure whether and how 0:0 relates to the well-defined case of 0:a (a≠0). There is no doubt that the pedagogical problem of dividing by zero revolves around three central issues: **0:a for a≠0** (equals zero), a:0 for a≠0 (undefined, since there is no value to satisfy the definition), and 0:0 (undefined, since there is no unique value to satisfy the definition).

Consequently *each of these issues should be treated separately in class, and subsequently inferences should be made to both the common and differentiating aspects of these problems.*

Moreover, this study indicates that the meaningful teaching and treatment of these issues requires some *basic knowledge concerning mathematical notions, such as, zero, division, and being undefined*, which are crucial to the comprehension of this perplexing topic. For instance, our findings strongly indicate students' unfamiliarity with the attributes of division, and in particular their lack of knowledge concerning the necessity of a *unique* solution, which is crucial for proving 0:0 to be undefined.

One of the most outstanding results has been, evidently, that preservice teachers did not find formal reasoning necessary for validating their mathematical assertions. One might argue that as long as the rules are perfectly implemented, there is no harm in pure memorization. But accepting mathematical rules as merely arbitrary might destroy the ability to reconstruct forgotten rules, and might even mix up visually similar cases with problematic ones, e.g., 0:5 and 5:0. Thus in mathematical explanations, *formalism should not be seen as a side issue, but as an important tool for clarification, validation and understanding* (Hanna, 1991). This particularly is relevant when dealing with complicated problems such as the case of division by zero.

References

Allinger, G. D. (1980). Johnny got a zero today. *Mathematics Teacher. 73*(3), 187-189.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education. 21*(2), 132-144.

Hanna, G. (1991). Mathematical proof. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 54-60). London: Kluwer.

Knifong, J. D., & Burton, G. M. (1980). Intuitive definitions for division with zero. *Mathematics Teacher. 73*,(3) pp. 179-186.

Reys, R. E. (1974). Division by zero: An area of needed research. *Arithmetic Teacher. 21*, pp. 153-157.

Wheeler, M., M., & Feghali, I. (1983). Much ado about nothing: Preservice elementary school teachers' concept of zero. *Journal for Research in Mathematics Education. 14*(3), 147-155.

CHILDREN'S INTERACTION AND FRACTION LEARNING IN A COMPUTER MICROWORLD: CONSTRUCTION OF THE ITERATIVE FRACTION SCHEME

RON TZUR

THE PENNSYLVANIA STATE UNIVERSITY

Interaction served as a theoretical core of a constructivist teaching experiment, in which two fourth graders' fraction learning was studied. Using a Task Oriented Approach to teacher-learner interaction (TOA), the teacher created and sustained learning environment in which both Linda and Jordan appeared to have challenging, meaningful, and enjoyable learning experiences. Teacher's tasks and constraints brought forth the children's equi-partitioning scheme and partitive fraction scheme and encouraged using iteration of various fractional parts to compose the whole or other fractional parts. Through interaction in a computer microworld, the teacher facilitated the children's construction of the Iterative Fraction Scheme (IFS).

A Constructivist Framework

The study addressed possible relations between children's interaction in a computer microworld and their fraction learning, because constructing fraction knowledge is a hurdle for many children and because research about teacher's involvement in children's fraction learning can hardly be found (Behr, Harel, Post, & Lesh, 1992). In the context of scheme theory (Piaget, 1980; Steffe, 1993; von Glasersfeld, 1989), I view children's learning as modifying and re-organizing current conceptual structures to neutralize perturbations. Schemes are conceptual structures by which the individual assimilates and/or organizes aspects of her or his experiences (Konold & Johnson, 1991). Schemes include three parts (von Glasersfeld, 1989): (a) recognition of a certain experience, (b) a specific activity that is related with that kind of experience, and (c) anticipation of a certain result that can turn into a prediction. Perturbation refers to any disturbance in a system produced by the scheme-based functioning of the system and it can be created through two types of interaction (Maturana, 1978): (a) interaction among elements within the child and (b) child-environment interaction.

Tzur(1995) specified three types of child-environment interaction to address possible involvement of the teacher in children's learning: child-initiated and self-referencing (e.g., mumbling to oneself), child-initiated and environment referencing (e.g., asking someone a question), and non-child initiated interaction (e.g., answering someone's question). In all three types, the teacher can assimilate the child's actions and language and interact with the child with an intention to engender specific perturbations that can lead the child to modify a specific scheme. The teacher may create constructive perturbations by engaging the child in goal-directed activities that bring about: (a) a failure to produce the anticipated result of the scheme, (b) a desirable result that was not anticipated and/or that was expected from another scheme, and (c) a blockage of the activity that should have resulted from the assimilating scheme (e.g., a constraint).

Perturbations are viewed necessary for learning but they can be debilitating when children cannot find paths of action to neutralize them (Steffe & Tzur, 1994). Children are likely to bear task-generated perturbations if they also experience the satisfaction of solving the task. Otherwise, children may be too frustrated, lack confidence, and lose interest. Thus, the teacher's role is to constitute learning situations in which children experience perturbations that are not so far removed from their current means of acting mathematically to impede the neutralization of those perturbations.

To engender constructive perturbations, Tzur (1995) proposed a task-oriented approach to teacher-learner interaction (TOA). In the TOA the teacher is involved in children's learning through posing and solving challenging mathematical tasks. The teacher, on the basis of her or his psychological, mathematical and pedagogical knowledge which includes an ever-changing model of the children's ever-changing mathematics, engages the children in posing and solving initial, reflective, and anticipatory tasks.

When the teacher poses an initial task the children need to: interpret it, set a goal, and initiate activities toward their goal. *According to the children's actions and language while solving (or posing) the initial task(s)*, the teacher poses reflective tasks—tasks that foster in the children re-presentations and abstractions of certain aspects of their experiences. For example, the teacher may ask children whether or not they had solved the task, or to explain/justify their answer. When the teacher interprets the children's work as indicating suitable ways of operating with initial and reflective tasks of a specific sort, the teacher poses anticipatory tasks. The purpose of anticipatory tasks is to challenge the children to work mentally rather than by carrying out actual activities.

Prior to constructing the IFS, Jordan and Linda constructed the equi-partitioning scheme and the partitive fraction scheme on the basis of four fundamental activities: decomposition, comparison, recomposition, and coordination with standard words (Tzur, 1995). In particular, those schemes were established on iterating fraction units to compose the whole and/or other fractional parts. Iteration was considered an appropriate means of composition because children use iteration of units to compose and operate with numbers. While working on sharing tasks, Jordan and Linda used iteration of a single part to create partitive units and used their numbers to coordinate those units with standard fraction words. For example, they called a part that was repeated 5 times to compose a specified 5-part whole "one-fifth," 3 such parts—"three-fifths," and the whole—"five-fifths." A major characteristic of the two first schemes was the children inability to conceptualize fractional parts that exceeded the whole (e.g., "ten-sevenths"). This paper addresses the children's work on tasks that led to overcoming those difficulties and constructing the IFS.

Method

A constructivist teaching experiment was conducted with Jordan and Linda during the second half of their fourth grade (1993). During that year we have conducted 29 videotaped teaching episodes once or twice a week, about 20-30 minutes each. Two faculty and 2-3 graduate students observed the teacher-researcher's work with the children and collaborated in the analysis/planning sessions between every two consecutive teaching episodes.

The teacher-researcher worked with the children in a computer microworld called Sticks. Sticks was dynamically developed to allow activities that seem to support conceptual operations needed for fraction learning. In this microworld a child may draw a stick using Draw and copy it as many times as desired using Copy. The child can mark a stick vertically using Marks, and any mark can be erased or moved. A child can partition any stick into *d* equal pieces by clicking on Parts, "dialing" a number (2-99), then clicking on a stick. Using Cut, the child can cut any stick at a desired point, join the stick to any other stick using Join, or position the stick at any desired place on the screen. A child can repeat a stick using Repeat, and break the stick into its pieces according to the marks on the stick using Break. Using Pull-Parts, the child can pull any number of parts out of a partitioned stick, and measure the stick by copying a reference stick into the ruler and using Measure. A child can also fill sticks (or parts) with 10 different colors using Fill, label them by a fraction symbol using Label, and cover or uncover sticks.

Establishing the Iterative Fraction Scheme (IFS)

Establishing the IFS required a qualitative transformation in the children's conception of part-to-whole relation that was constructed in the equi-partitioning and the partitive fraction schemes (Tzur, 1995). In those two schemes, the partitioned whole was constructed and conceived as containing *exactly d* parts and the child could not iterate partitive units over its boundaries. If the number of iterations exceeded the original whole (e.g., iterating 1/8-stick ten times), the child regarded the result as a new partitioned whole (10/10-stick), not as a 10/8-stick.

To overcome the stumbling block of the bounded whole, the teacher's goal was to capitalize on the children's use of iteration of fractional units to create and make sense of fractional parts that exceed the whole. The children needed to learn to *maintain the original ratio between the part and the whole regardless of how many times the fraction unit was iterated*. To do so, the teacher posed tasks of iterating fractional parts within, and then over the boundaries of the whole. As a result, Jordan and Linda re-constructed the fractional parts as *iterable fraction units*, i.e., conceptual structures prior to iterating them.

Iterating Fraction Units Over the Boundaries of the Whole

On 3-31-93, the teacher suggested to play "Guess the stick I am thinking of" only with elevenths. Linda posed the first problem (3/11) and Jordan partitioned the whole into 11 parts, then filled 3 of them in a different color. After the teacher asked Jordan, "You want to pull it out" he pulled 1/11 and repeated it three times. Next, Jordan "thought of 6/11" and Linda solved it by partitioning a new stick into 11 parts, filling 6 in a different color, and pulling the parts out of the whole.

The children's actions indicated a dual conception of the fractional parts (3/11 and 6/11)—units embedded within, or disembedded from the partitioned whole. This dual conception served as a precursor to iterating such a unit over the boundaries of the whole while maintaining the 1-to-11 part-to-whole relation. To maintain the 1-to-11 relation even if the number of such parts exceeds the 11/11-stick, the child must abstract the 1/11 and conceive it as a unit in spite of its location or perceptual appearance.

The teacher asked, "I am thinking of a stick that is twice as long as the 3/11," and Jordan copied the 3/11 twice and joined the copies. Jordan's immediate use of Copy and Join indicated that he was able to envision an abstract stick like the stick that the teacher "thought of" and to interpret the task as implying iteration of the 3/11-stick twice. The teacher asked Jordan how much was the joined stick of the original whole. Jordan said "6/11," and explained: "I know that 3 plus 3 is six, and, then, eleven pieces of it." Jordan's explanation indicated that he did *not* confuse the 1-to-11 relation with the number of times the 1/11-stick or the 3/11-stick were iterated. Jordan seemed to feel the logical necessity regarding the type of part (1/11) each of the 6 pieces were. In this sense, he was ready to solve a task of iterating a fractional part over the boundaries of the whole. At this point, Linda smiled and asked Jordan: "I think of a stick that is 11/11 of that [original] stick." Jordan partitioned a copy of the whole into 11 parts and Linda said "Yeah." Next, the teacher asked, "we want it out of..." and "can you name it another way?" and Jordan copied the 11/11-stick and answered: "A whole."

Linda's choice of 11/11 and her smile seemed to reflect her current conception of the bounded whole. It indicated that the "largest" number possible is 11/11, and she realized the meaning of such a number—pulling the whole out of itself. The teacher could have not asked for a better choice, since Linda's 11/11 stressed the 1-to-11 relation between the part iterated (1/11) and the stick she "thought of." The teacher's question for a different name and Jordan's answer "A whole" supported both children's awareness of the 11/11 partitioned whole.

Since Linda used the "largest" number of elevenths possible, Jordan thought "of a stick that is 10/11," and Linda solved it. Then, it was the teacher's turn to pose a task.

Protocol 1 (3-31-93, episode 21)

T: OK, now I'm thinking of a stick that is twice as long as the 6/11.

L: (Copies the 6/11-stick, repeats it twice, and puts next to the 6/11.) (cf. Figure 1)

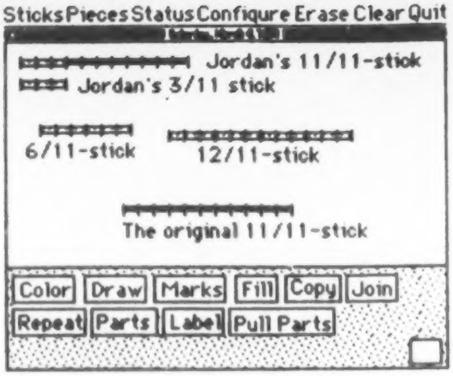


Figure 1. Linda repeated the 6/11-stick twice.

T: I have a question. Is that the original one (points to the original 11/11-stick)?

Both: (Nodding yes.)

T: Oh, OK. How much is that one [12/11], that you made right now...

J: (Interrupts the teacher's talk) Twice as long as that one.

T: ... of the original one?

J: Oh ...

L: There's only one left over from this one (points to the 11/11-stick, probably means one piece more than the whole).

J: (Hesitantly) There's ... umm, see there's eleven, there's 12 pieces, and, umm, people that came to the party, and they eat 11, so there's one left, so one more on it.

T: (Nodding yes) So how much is it?

J: (Hesitantly) So it's eleve ... twelve-elevenths?

T: (To Jordan) Twelve-elevenths you say? (To Linda) What do you say?

L: (Shrugs her shoulders) I don't know.

T: (To Jordan) How did you figure it out?

J: (Points to the screen with confidence) There's 6 ..

L: There's 12 pieces ...

J: There's 6 here, and 6 plus 6 is 12, and there's 11 here (points to the original 11/11-stick).

T: (To Linda) What do you say? Did you understand what he said?

L: Yes.

The teacher chose the unit to be iterated (6/11) and the number of iterations (2) to foster a specific perturbation that the children could neutralize. Specifically, we hypothesized that Jordan and/or Linda would be able to use their doubling strategy to make sense of the result. Additionally, this result exceeded the reference whole in only one part. In this way, the teacher hoped to minimize the level of computational difficulty involved in the complex enough conceptual transformation. As it turned out, the choice helped the children in neutralizing the perturbation of a fractional part that contained more parts than the 11/11-stick.

I suggest that the key to the transformation in the children's conception was the sequence of interaction in which they posed and solved tasks that involved iterating fractional parts. Particularly, they experienced the iteration of a fractional part (3/11) twice and explicitly clarified the boundaries of the 11/11 partitioned whole. In this context, *they* produced the "impossible" 12/11 by iterating a fractional part (6/11) twice rather than by iterating 1/11 twelve times.

Since Jordan was the one who "doubled" the 3 parts of 1/11, it was not surprising that he also suggested the solution of 12/11. Jordan's explanation clearly indicated that he solved the problem with 6/11 using the experience of doubling; his hesitation when he finally said "So its eleve ... twelve-elevenths?" indicated his perturbation. The reason for this perturbation was twofold: (a) Jordan was not aware of the significance of obtaining 12 pieces prior to iterating the 6/11 and (b) he realized there were 12 pieces as a result of his action. Jordan's perturbation indicated a conflict between his prior notion of the "impossibility" of fractional parts such as 12/11 and the result of his action (doubling the 6/11).

Jordan's independent use of the birthday cake context illustrated the essential role of an everyday context to his capability of thinking and communicating mathematically. This context helped him to re-present a situation in which the 1-to-11 relation could be distinguished from the number of parts resulting of the doubling operation. Linda's reaction indicated that she was also able to re-present the situation and follow Jordan's distinction—she explicitly referred to "12 pieces" and seemed to understand that each piece was 1/11.

Linda's solution ("There's only one left over from this one") and Jordan's solutions (12/11) appeared to reflect different foci. Unfortunately, the teacher did not understand Linda's answer at the time. It seems that she neutralized the perturbation by focusing on the 11/11 partitioned whole and the remainder 1/11 because she previously posed the problem of making 11/11 to Jordan. In this sense, Linda's solution can be characterized as a whole-based focus. Linda's solution suggested that she was able to maintain both the 1-to-11 relation and the partitioned 11/11-stick. Specifically, she was able to decompose the 12/11-stick into two units—11/11 and 1/11, which highlighted her use of the 1/11 as an iterable unit similarly to her use of one when producing numbers (i.e., 12 can be made of 11 and 1).

To see if Linda could independently conceptualize a fractional part that exceeds the whole, the teacher asked Jordan to make a stick that is three times as long as the 6/11 and he did. Then, in response to the teacher's reflective questions "How much is that one?" and "How did you know that?" Linda thought for about 6 seconds, said "Eighteen-elevenths," and explained: "Because ... 6 times 3 is ... 18." Next, the teacher posed an anticipatory question, "If I would ask about 5 times as long as the 6/11?" and both children answered that it would result in "thirty-elevenths."

Explaining the Activities of Another Child. The teacher asked Jordan to iterate the 6/11 because it was Jordan's turn, and because the teacher wanted to switch the roles in the previous task. Asking a child to explain the other child's activities of solving a task in the microworld greatly supported the children's cooperation and their use of mental operations. To explain the other child's iteration activity, the "explainer" must assimilate, re-present, and make sense of the "iterator's" activities. Likewise, the "iterator" assimilates and interprets the "explainer's" ideas by relating them to his or her own activities.

The way Linda explained Jordan's activity highlighted her clear distinction between the generating fraction unit (1/11) and the number of times it was iterated. Specifically, Linda was able to reflect on Jordan's activities and create 18/11 as a unit of units of units (three units of 6 units of 1/11) while using her notion of multiplication. In this sense, Linda found the result of iterating a fractional part in conjunction with multiplication similarly to her activities with numbers. Linda's use of multiplication contributed to the children's construction of the IFS. It opened the way for using a familiar operation to deal with fractional units and strengthened the children's capability to maintain the part-to-whole relation. For example, the children easily answered the teacher's next question (5 times as long as the 6/11) that fostered *predicting* the potential result of an activity without carrying it out.

In the next episode we used a "naming" activity to further enhance the children's IFS. This time, we began with iterating a fraction unit (1/8) nine times. The children appeared to be immersed in the lengthy naming activity and challenged by the goal of producing an appropriate fraction name. Eventually, they summarized their understanding of the iterated unit as both "Nine-eighths" and "One and one-eighth."

Discussion

The children's establishment of the IFS occurred in an activity of iterating fractional part over the boundaries of the whole. To make sense and communicate their conceptualization, the children also had to coordinate their new conception with a standard word as part of the construction process. This coordination served as a goal towards which they worked (i.e., explaining to others), as well as a means to conceptualize the part-to-whole relation. In this sense, the "naming" activity served the dual function of constructing concepts—conceptualizing and communicating.

"Guess the stick I am thinking of" was a teacher-initiated task that fostered child-sustained interactions. The children were engaged in a playful mathematical interaction in which they posed and solved tasks, and justified the solutions. Thus, the teacher was able to pull back, to lessen control over the situation, to observe the children's work, and to ask appropriate reflective or anticipatory questions. In this sense, a task that fosters child-sustained interaction supports the teacher's involvement in the children's learning while increasing their ownership over the learning situation. Additionally, a child-sustained interaction creates an open learning situation. After the teacher stated the task, one could not predict which child would pose the first task, nor what task would be posed or what computer action would be used. It seems that the open learning situation strengthened the children's conception of oneself and of one another as a mathematical source, that is, as "a person.... consulted for information or providing initial inspiration" (New Webster's Dictionary, 1993, p. 948).

Through the child-sustained interactions of naming and playing "Guess the stick I am thinking of," Linda and Jordan transformed their conception of the bounded whole and part-to-whole relation. This transformation was based upon iteration—the very operation used in producing fractional parts. Psychologically, solving tasks in which a composite fraction unit is iterated within the boundaries of the whole (3/11 → 6/11) and then over the boundaries (6/11 → 12/11) makes sense. Ordering the tasks that way facilitated the children's focus upon the relation among *parts of the same sort*. Implicitly, the generating unit (1/11) was taken as a given and the children concentrated on the results of iterating it and producing units of units of units (e.g., 2 units of 6 units of 1/11). As Jordan's first explanation of this operation indicated, iterating a composite partitive unit resulted in a distinction between the generating fraction unit and the number of iterations. This distinction brought forth a transformation in the children's conception of part-to-whole relation. Hereafter, they were able to maintain the 1-to-11 relation and the partitioned original whole and experience no difficulty in conceptualizing and operating with fractional parts that exceed the whole.

References

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 296-333). New York: Macmillan.

Konold, C., & Johnson, D. (1991). Philosophical and psychological aspects of constructivism. In L. P. Steffe (Ed.) Epistemological foundations of mathematical experience (pp. 1-13). New York: Springer-Verlag.

Maturana, H. R. (1978). Biology of language: the epistemology of reality. In G. A. Miller and E. Lenneberg (Eds.) Psychology and biology of language and thought: essays in honor of Eric Lenneberg (pp. 27-63). New York: Academic.

New Webster's Dictionary and Thesaurus of the English Language (1993). Danbury, CT: Lexicon.

Piaget, J. (1980). The psychogenesis of knowledge and its epistemological significance. In M. Piattelli-Palmarini (Ed.), Language and learning: The debate between Jean Piaget and Noam Chomsky (pp. 23-34). Cambridge: Harvard University.

Steffe, L. P. (1993). Children's construction of iterative fraction schemes. A paper presented at the 71st annual meeting of the National Council of Teachers of Mathematics. Seattle, Washington.

Steffe, L. P., & Tzur, R. (1994). Interaction and children's mathematics. Journal of Research in Childhood and Education, 8, 2, 99-116.

Tzur, R. (1995). Interaction and children's fraction learning. Unpublished doctoral dissertation. Athens, GA: The University of Georgia.

Von Glasersfeld, E. (1989). Cognition, construction of knowledge, and teaching. Synthese: a special issue of philosophy of Science and Education.

Precalculus and Graphic Calculators: The Influence on Teacher's Beliefs*

Paola Valero and **Cristina Gómez**

"una empresa docente" - Universidad de los Andes (Colombia)

This study explored the effects on the belief system with respect to mathematics, its educational aims, its teaching and learning and the role of instructional materials, of a teacher who was involved in a curricular innovation centered on graphic calculators. A Beliefs System Typology including five different ideologies about mathematics education was used as a conceptual framework. Three different research techniques were applied to the teacher before and during the introduction of the curricular innovation. The evidence shows that the teacher modified her behavior in class, but she could not change completely her beliefs system. Being involved in the experience began a questioning process which could lead to a real eventual change in her beliefs system.

Introduction

This paper reports on an empirical, exploratory study framed within a larger research program aiming at the examination of a series of changes in some elements of the curriculum when graphic calculators are introduced in a Precalculus course for first-year university students.The main aim of this study was to explore the influence of the introduction of graphic calculators within the classroom as a part of a whole curricular change, on the teacher's beliefs with respect to mathematics, the aims of mathematics education, its teaching, learning and the role of instructional resources.

Several studies have shown the importance of investigating teacher's beliefs due to their close link to instructional practice (Thompson, 1984, 1992; Fernandes, 1995). Special emphasis is also made in the relationship between teachers' beliefs and processes of change (Pehkonen, 1995). Reported evidence shows that the teacher's model of mathematics, its teaching and learning greatly influences his/her behavior. This is the reason supporting the importance of looking at the teacher's beliefs system when he/she is involved in a change like the one introducing graphic calculators as a tool for mathematical knowledge construction within a classroom.

In spite of having noticed this special link between beliefs and behavior, most of the research conducted has mainly used research techniques and instruments that focus on "what the teacher says" as a way to deduce what he/she actually thinks. The present research used a combination of different research instruments targeted to explore both what the teacher does and says to conclude about what he/she thinks. Besides, most of the research use a deductive methodology that analyzes either some cases or ample samples of teachers and, as a result, build a typology of beliefs. This

---

* This research was supported by the Colombian Institute for the Development of Science and Technology (COLCIENCIAS), the Foundation for the Development of Science and Technology of the Colombian Central Bank, the PLACEM project, and Texas Instruments.

research used a pre–defined typology and identified the teacher's position on it.

Conceptual framework

A belief system is a structured set of views, or groups of views, conceptions, values or ideologies held by a teacher with respect to the elements composing his/her professional teaching practice. This system may have different degrees of intensity, i.e. some conceptions are stronger than others; it is supposed to be coherent, but the groups of beliefs may not necessarily be interdependent, i.e. some conceptions may be isolated from the rest of them; and it is dynamic because it gets questioned whenever the teacher is involved in a nourishing practice, in other words, there is a constant feedback among beliefs and experience. Although there is change and questioning, these are slow, long-term processes.

Among the several elements which can be considered in a beliefs system, the five most relevant for the research were chosen. These five elements –teacher's view about mathematics, about the aims of mathematics teaching, about learning, about teaching and about the role of instructional materials– may be seen in relation to different positions with respect to all of them. Following Ernest's typology (1991), five different kinds of teachers, with their corresponding belief systems, may be sketched:

	Industrial Trainer	**Technological Pragmatist**	**Old Humanist**	**Progressive Educator**	**Public Educator**
Mathematics	Set of truths and rules associated with authority	Body of unquestionable, useful knowledge	Body of pure, structured knowledge	Set of structured, personalized knowledge	Set of socially built knowledge that may be changed
Aims of M.E.	Mechanization of basic skills	Usefulness of knowledge. Application to technology	Transmission of cultural and rational values. Mental formation	Individual development and self-realization through mathematics	Individual potential development, aimed at social change
Teaching	Transmission of skills, repetition of exercises	Instruction on the mastering of skills. Applied problem-solving	Explanations, motivation and transmission of structures	Promotion of personal learning	Discussion, investigation, questioning
Learning	Authority, memorization, mechanization, rote-learning	Practice and application of skills	Understanding of mathematical structures and their application	Investigation, autonomy, creativity, mathematics games and exploration	Internalization of social constructions of mathematics. Everyday life problem-solving
Materials	Paper and pencil. Anti-calculators	Materials promoting experimentation. Calculators and computers are allowed	Traditional materials. The minimum necessary materials are allowed	Any kind of material that enhaces the formation of concepts and representations	Different kinds of materials. Each student uses those with which he/she fees comfortable

Since teacher's beliefs are cognitive structures and, therefore, exist in the realm of "what he thinks", and this realm can not be observed directly, it is necessary to approach them through another two realms: The realm of "what the teacher does" — his/her behavior in actual teaching practice with students—, and the realm of "what the teacher says" (his/her expressions and opinions about what is on his/her mind). This distinction is quite important because inconsistencies between those realms may occur: A teacher saying he hold a social–constructivist view of mathematics, its teaching and learning, may act as an industrial trainer when interacting with students. The question raised here is whether what he expresses reflects in reality what he thinks. Due to the presence of these different realms and the possible conflicts between them, exploring beliefs imply using more than one research technique in order to allow a contrast between information as a clear idea of the real conceptions held by the teacher.

Methodology

The research was conducted during two semesters, one previous to the introduction of the curricular change, and the other with the curricular change. The same in-service university teacher was observed during both semesters. Students in both groups shared similar characteristics. Three different techniques were applied during both semesters in order to establish their results for each one of the periods considered and then, contrast them for finding possible changes in the teacher's beliefs system. The techniques used were:

Classroom Observations. A video recording of the teacher's performance in the classroom in both semesters was taken during the length of the teaching-sequence for quadratic functions. A representative sample of the videos were analyzed with respect to five variables –teacher's usage of verbal and no-verbal language, teacher-student interaction patterns, teacher's timing, kind of exercises and questions proposed by the teacher, and teacher's treatment of students' error. These variables, as noticed by some researchers (Robert et Robinet, 1989), differ from one teacher to another when they hold different beliefs.The variables were operationalized through a detailed description of the behaviors of each type of teacher in the Beliefs System Typology with respect to them. The specific instruments used for the analysis of the videos were the Classroom Interaction Analysis proposed by Amidon (1971) and a content-analysis technique based on the definition of the variables mentioned above. These instruments gave information about "what the teacher does", in other words, about how the teacher's behaviors reflect her beliefs with respect to the teaching and learning of mathematics and the role of instructional materials.

Beliefs Instrument. The Likert-scale, beliefs instrument designed and validated by Ibrahim (1990) was translated into Spanish and adapted to be applied to the population of mathematics teachers to which the teacher observed belongs. This test was applied twice during both semesters in order to verify the consistency of her answers. This test gave information about the teacher's position with respect to mathematics and the aims of mathematics education and was centered in "what the teacher says".

Guided interviews. The interviews held with the teacher were developed during both semesters with respect to, on the one hand, a sample of the videos for each semester and, on the other hand, her answers to the beliefs instrument. The teacher was asked to comment on her behavior in the videos and her divergent answers in the test. The information obtained with this instrument was used as a means to contrast and validate the evidence found through the instruments mentioned above.

Data and Results

Classroom Observation

For the semester previous to the introduction of the curricular innovation based on the graphic calculators, the teacher dominated the interaction in the classroom. She talked the majority of the time (67%) an her interventions were centered on the presentation of mathematical content to the students. The student's interventions took 15% of the time and were mainly predictable answers to a teacher's question. Few self- initiative and participation was observed in the group of students. The teacher used to talk during long periods of time and established a kind of monologue because she asked and immediately answered her own questions. Besides, when a student was on the blackboard solving an exercise, she guided the student, corrected his/her mistakes and explained what was done. The mathematical content of the class was worked through the teacher's explanations or the solving of mechanical exercises proposed in the textbook for homework.

In the semester where graphic calculators were introduced, the teacher's interventions were reduced to 57% and the student's spontaneous participation increased. Dialogues between the teacher and one student, the teacher and a group of students or among the students were developed with respect to the mathematical content of the class. The knowledge was presented through the performance of students on the blackboard when solving the exercises proposed by the textbook. The students sometimes presented a variation of the original exercise in order to evidence a difficulty they had in the comprehension of a concept. The teacher allowed the students to go on with their mathematical discourse, in spite of having made a mistake. She let them realize their fault or asked questions addressing the point where the mistake was made. She emphasized the meaning and use of mathematical wording. This reflected a concern of the teacher for helping the students understand the concepts, but she did not propose different activities or problems to actually enhace their understanding.

From the evidence shown above, it can be said that there were significant changes with respect to the teacher's behavior in the classroom. Whereas during the first semester she dominated the interaction with the students and the latter had little participation, during the second one the students leaded the class with their interventions, questions and doubts. There was also a change in the language used in class: while the language used during the first semester emphasized the mechanical aspects of the procedures and algorithms involved in the concepts, the language used during the second one focuses on the reasons why a given procedure was used, the concepts and the relation between them. Although there was not a real change in the exercises proposed by the teacher to be solved in class, there was a change in the way she treated the errors made by the students. In the first semester the teacher used to guide the student to an extent that he/she was not allowed or did not have the opportunity to make a mistake. On the contrary, during the second semester, she allowed mistakes and built new knowledge on them. The students participated in the correction of the error and impose their pace of work. This confirms Gómez and Rico's results on social interaction and mathematical discourse research with the same teacher and group of students (Gómez and Rico, 1995).

Beliefs Instrument

The 60 items test was applied to a sample of 33 from the total population of 58 teachers of the Department of Mathematics of the Universidad de los Andes. Factor analysis of the answers determined five factors: The first factor grouped 8 items explaining 14.9% of the variance. These items describe mathematics as a body of truths made of logical steps not necessarily mechanical, and related to concrete things. The second factor included 10 items explaining 12.1% of the variance. These items depict mathematics as a set of universal rules and laws and an activity where everything must be demonstrated. The third factor gathers 8 items explaining 10.3 of the variance. Mathematics in this factor are a set of truths whose validity and development depends on people. The fourth factor joins together 4 items explaining 9.5 of the variance. These items describe mathematics as an activity with practical applications, where some basic skills are required. And the fifth factor is composed of 5 items explaining 7.6% of the variance. The items describe mathematics as an human created activity where knowledge is being built. Mathematics furnishes people with mental structures. These five factors were associated with the five views of mathematics stated in the Beliefs System Typology.

This factor groups were taken as the reference to compare the answers given by the teacher to the test in both semesters. As a result of the comparison of the teacher's answers with respect to the answers obtained from the sample, a number representing her position within each factor was obtained. A paired proof $(p=0.05)$ of these values showed that there was no significant difference between them. Therefore, no changes in the beliefs of the teacher with respect to mathematics and the aims of mathematics education were observed.

Factor / Semester	F1 Progressive educator	F2 Industrial trainer	F3 Technological pragmatist	F4 Old humanist	F5 Public educator
First	1.090	6.171	1.372	4.014	-5.188
Second	1.939	3.339	1.933	4.938	-4.400

Guided interviews

The data collected in the interviews supported the findings presented above. With respect to her change in behavior during the second semester, the teacher expressed surprise for having behaved in a way she was not used to. Nonetheless, her explanations to this fact relied on external factors such as the group of students. Some reasons for her behavior were related with her interaction with the researchers who were also teaching the same Precalculus course and experimenting with calculators. She clearly noticed a difference in the dynamic of the class, motivated by the graphic calculator. The following sentences support these ideas:

> *R: During a meeting with the researchers you expressed a surprise in relation to the students' participation in class. Do you thing that such a participation was caused by something you did or because their were participative?*
> *T: I think there were two reasons. On the one hand, he human part, I mean, the group itself was participative and that is very relevant. On the other hand, I have never used this group work system. One usually makes students work on the blackboard and in small groups during workshops, but that is not permanent. But when they frequently work in groups, they know each other and, therefore, they interact even more. Nonetheless, this active interaction and participation is impossible if they are not participative.*
> *R: And calculators, did they have a part on that interaction?*
> *T: I would say that yes. Yes, they help, and I wish they helped more. Why do they help? First, because they give the students an incredible security [...] They trust the calculator or question the teacher more than what they were used to, when the calculator shows them an unexpected result. I would say that the calculator changed my thinking a lot and this change was obviously supported by the methodology one has to follow. I mean, I was an open enemy of calculators in the classroom, and in part this experience confirmed what I thought. In order to make calculators a really useful tool, the whole class system has to change. Every thing, the methodology, the exercises, the role of the teacher. [...] Because if I teach in exactly the same way I used to, calculators worth nothing.*

In spite of having reflected this changes in behavior associated to her views on the teaching, learning and role of instructional materials, the interviews on the beliefs instrument sustained the lack of change with respect to her beliefs about mathematics and the aims of mathematical education. The teacher was unable to clearly support the differences in the answers and, when she was directly asked about the nature of mathematics, she answered:

> *R: So you agree on this, that mathematics are mainly assumptions, laws and general rules...*
> *T: Suppositions, yes, they have something of that. Mathematics have to have assumptions, laws, rules. But I would not say they are only that. Now, why do mathematics exist? Well, I would say mathematics almost give, oh yes, they give an order, like a justification; they are the method that was found, let's call it that way, to give an order, a justification, some proofs and that is what has organized, ruled the situations we live [...] I think mathematics are the only way and by now no other has been found to give an ordered organized structure to go from one stage to another.*

Interpretation

The results presented in the previous section with respect to the beliefs system of the teacher observed show that during the first semester she was clearly sharing the position of the "industrial trainer". Her beliefs with respect to mathematics and the aims of mathematics education were associated with a dualist position in which mathematics are a set of unquestionable, accepted truths and that mathematics education aims at the mechanization of basic skills. Her behavior also reflected a sharing of this position because of her insistence on assigning repetitive exercises to train the students with. She definitely rejected calculators.

In the second semester, there was an alteration in her beliefs system. This alteration may not be considered as a real change of her belief system because, despite her changes in behavior, her beliefs about mathematics did not suffer a modification. With respect to this last point, she continued sharing an "industrial trainer" position. But her behavior tended to be that of the "public educator". This fact and her perception of her own behavior show that the curricular innovation based on the graphic calculators altered her way of interacting with students, but was not enough to change her beliefs as a whole, at least during this short period of time.

Conclusions

In general, the present research showed that the participation of the teacher in the curricular innovation involving graphic calculators induced some modifications in what she did and said, but no significant changes in what she thought. The continuity of her "industrial trainer" position about mathematics, its educational aims, and the shift towards a more constructivist, relativist view of the learning, teaching and materials reflect a "destabilization" on her beliefs system, but not a real change. Besides, a deep transformation in cognitive structures requires more time to be achieved. And change is even slower when those structures are unconscious.

The most significant achievement reached by the teacher was the beginning of a questioning concerning her beliefs. During the six months of the curricular innovation, she began to be aware of the possible approaches to the teaching and learning of mathematics. One year after the experience, she clearly revealed a different and more coherent perception of her professional practice. On the one hand, her comments on mathematics teaching and learning as well as the activities she proposes to her students in all her courses are closer to a problem–solving, constructivist view. On the other hand, she has been studying both theoretical and research papers on mathematics education and seems to be concerned about the knowledge of this discipline.

Discussion

A final discussion about the issues of teacher's beliefs change in relation to the curricular change involving graphic calculators has to highlight the following main points. First of all, technology itself does not promote change. The dynamics that the new curriculum imposed in the traditional development of the class helped the destabilization of the teacher's beliefs. The innovation implied the use of small group work, which, in itself, dynamizes students' interaction and participation. Besides, the autonomy that the calculators give to students with respect to the teacher's authority for "having the truth" makes a shift in the way they typically view the teacher. Second, the questioning is the result of the conflict arising from the contrast between the teacher's beliefs and real experience. As some basic elements of the curricular system were altered, she had to accommodate her behaviors to the new situation. This accommodation generated a feeling of discomfort which lead her to question her traditional way of behaving. Third, this questioning also dealt with her knowledge about some issues of mathematics education for finding answers to her doubts. She manifested her lack of didactical knowledge:

> *T: [In class] I do what I knew when I made my teaching practicum here in the University, a long while ago. I mean, the only moment I had a guidance was when I was a student. [...] I graduated, imagine, several years ago and I still have the same teaching system. Things*

> *change. I mean, I didn't give classes nor touch a book for 10 years. I assume many things and my reality is that I dash against my students when I realize my assumptions are not valid any more.*

However, she perceived her practice could have a different support and asked for specialized assistance from the researchers. Finally, the process of beliefs change is slow and requires time to reach stability, as well as a constant motivation for the teacher to keep evolving through a permanent exchange between experience and beliefs.

References

Amidon, P. (1971) *Nonverbal Interaction Analysis. A method of systematically observing and recording nonverbal behavior.* Minneapolis: Paul S. Amidon and Associates.

Ernest, P. (1991). *The Philosophy of Mathematics Education.* London: The Falmer Press, 1991.pp. 603-612.

Fernandes, D. (1995). Analyzing four preservice teachers' knowledge and thoughts through their biographical histories. In Meira, L. & Carraher, D. (Eds.) *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* Recife: Universidade Federal de Pernambuco, pp. 162-169 (Vol. 2)

Gómez, P., Rico, L. (1995). Social interaction and mathematical discourse in the classroom. In Meira, L., Carraher, D. *Proceedings of the 19th PME Conference.* Recife: Universidade Federal de Pernambuco, p. I-205.

Ibrahim, Hashem Ibrahim. (1990). *A multidimensional mathematics belief instrument with content and construct validity and its application to elementary and secondary preservice teachers.* Ann Arbor: UMI Dissertation Services, 1993.

Pehkonen, E. (1995). What are the key factors for mathematics teachers to change. In Meira, L. & Carraher, D. (Eds.) *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (Vol 2, pp. 162-169). Recife: Universidade Federal de Pernambuco.

Robert, A. et Robinet, J. (1989). Représentations des enseignants de mathématiques sur les mathématiques et leur enseignement. *Cahier de DIDIREM,* N° 1. Paris: IREM Paris VII.

Thompson, A.G. (1984). The relationship of teacher's conceptions of mathematics teaching to instructional practice. *Educational Studies in Mathematics.* 15, pp. 105-127.

Thompson, A.G. (1992). Teacher's Beliefs and Conceptions: A Synthesis of the Research. En Grouws, D.A. *Handbook of Research on Mathematics Teaching and Learning.* New York: Macmillan, pp. 127-146.

CONDITIONAL PROBABILITY AND THE LEVEL OF SIGNIFICANCE IN THE TESTS OF HYPOTHESES

A. Vallecillos and C. Batanero

University of Granada, Spain

ABSTRACT

In this paper we present a study of university students' errors concerning the significance level in statistical tests, to which much research work has been dedicated both in education and experimental methodology. In depth interviews with selected students show the persistence of conceptual errors concerning the level of significance in students with a good understanding of conditional probabilities. These errors seem to be produced by the "illusion of probabilistic proof by contradiction" (Falk and Greenbaum, 1995) as well as by students' misunderstanding of some key concepts in the tests of hypotheses.

RESUMEN

En este trabajo presentamos un estudio realizado sobre errores referidos al nivel de significación en un test de hipótesis, tema al que se ha dedicado mucha investigación en educación y metodología experimental. Una serie de entrevistas en profundidad realizadas con una muestra seleccionada de estudiantes universitarios pone de manifiesto la persistencia de errores conceptuales sobre este tema en estudiantes con una buena comprensión de la probabilidad condicional. Estos errores parecen ser producidos por la "ilusión de la prueba probabilística por reducción al absurdo" (Falk y Greenbaum, 1995), asi como por los errores conceptuales de los estudiantes sobre conceptos claves en el test de hipótesis.

BACKGROUND

Hypotheses testing is a major methodological tool for decision-making and for research in different sciences. However, many authors have described errors in the use of this technique, and, due to the risk of reaching invalid conclusions, have criticized its use (as a summary, see Morrison and Henkel 1970). One key aspect in the logic of hypothesis testing is the significance level, which is defined as the "probability of rejecting a null hypothesis, given that it is true". That is, α=P(reject $H_0|H_0$ is true). A particular misconception, consisting in the interchange of the events in the conditional probability defining the significance level has been widely described (Birnbaum, 1982; Falk 1986; Vallecillos and col., 1992). This error consists in the mistaken interpretation of the level of significance as "the probability that the null hypothesis is true, once the decision to reject it has been taken", that is, in interpreting α as $P(H_0 \text{ true} | H_0 \text{ rejected})$. Difficulties in discrimination between the "two directions" of conditional probabilities have been recognized by Diaconis and Freedman (1981), who labeled this misinterpretation as *"the fallacy of the transposed conditional"*. Falk (1986) suggested that, in the particular case of significance level, the verbal ambigüity in the term "Type I error" may also provoke confusion between the two opposite directions of the conditional probabilities among the students, who may believe that they are dealing with the probability of a single event. According to Menon (1993), the phrase "Type I error" induces the idea of only one event and, therefore, people tend to forget that they are dealing with a conditional probability, which necessarily involves two events. This induces error when interpreting the significance level in terms of the conjunction of the two events, "the null hypothesis is true" and "the null hypothesis is rejected" in either of the following ways: 1) The null hypothesis is true and it is subsequently rejected; or 2) The null hypothesis is rejected and it is subsequently found to be true.

Falk and Greenbaum (1995) present a critique of the logical structure of statistical tests, analyzing the possible causes of the persistence in using the significance test, in spite of the misconceptions that have been described. They suppose this is due to the *"illusion of probabilistic proof by contradiction"*, based on a misleading generalization from logical reasoning to statistical inference. This generalization may be explained by the seemingly parallel arguments in proof by reduction ad absurdum and that of rejection of the null hypothesis. Falk and Greenbaum attribute the prevalence of this illusion to the following intrinsic cognitive mechanism: similarity of the reasoning in statistical tests to modus tollens reasoning; verbal ambigüity in describing the type of errors associated with the test of hypotheses and the wish of researchers to rule out chance.

RESEARCH PROBLEM AND METHODOLOGY

The main objective of our study was to deepen in the analysis of the errors referring to the interpretation of the level of significance and its relationships with the understanding of conditional probabilities. It is part of a wider study including a larger sample of students and questions described in Vallecillos (1994). Seven university students were asked to complete the questionnaire included as an App endix. The students were taking a major in Medicine at the University of Granada and had studied Statistics for a complete year. Their participation in our research was voluntary and we asked them to carefully study the test of hypotheses beforehand. They where selected among the better students in their group. Once they completed the questionnaire, each student was interviewed for about half an hour. In particular, we questioned the students about their understanding of conditional probabilities, the possibility of computing the probability of a hypothesis as a result of a test and the procedure required to do this computation. Another point of interest was the assessment of the students' discrimination between the statements in items 1 and 2 in which Birnbaum(1982) has reported lack of discrimination by the students. We also investigated whether the students considered one of these statements to be equivalent to the definition of the significance level they were taught. Finally we suggested the students to compare the two probabilities involved in paragraphs a) and b) in the medical diagnosis problem and to describe the similarities of this situation with the test of a hypothesis.

RESULTS AND ANALYSIS

Interpretation of conditional probabilities

All the subjects, except Student 7, correctly solved the diagnosis problem and none of them had difficulty when interpreting the conditional probabilities in questions a) and b) of this problem: "*They are just the reverses; They mainly differ in the conditioning event, in the sense that, in the first question, the conditioning event is to be suffering from cancer and in the second,.... the test being positive*" (Student 5). They were also able to explain and to present correct and meaningful examples of the difference between an ordinary probability and a conditional probability: "*For example, an illness and a risk factor to assess whether this illness* *is influenced or not by this risk factor, this would be a conditional probability. A simple probability would be the incidence of a given illness in a population"* (Student 7). They didn't consider the existence of only one event in the expression "probability of Type I error", and they were conscious that in a conditional probability there are two different events involved: "*Only one event? I don't understand you. Here there are a conditioning event and another event that is conditioned"* (Student 5).

In general our students could interpret the diagnosis problem from a hypothesis testing approach: "*Error α, the probability of accepting that a person has cancer when he/she has no cancer, when the hypothesis of having no cancer is true"*. (Student 3). "*I should identify the false negatives with error β"* (Student 6). Moreover, this student wrote: "$\alpha = P(H_1|H_0) = P(H_1 \cap H_0) / P(H_0)$", when he was asked to compare his answer to question c) of the problem with the possible errors in hypothesis testing. Consequently, he not only realized that he was dealing with a conditional probability, but he wrote the general formula to compute this probability.

To sum up, our students did not believe they were dealing with only one event, neither they wrongly interpreted the conditional probability involved in the medical diagnosis problem. Moreover, they were able to correctly compare these probabilities with the definition of the significance level. Only Student 4 showed some difficulty related to verbal ambigüity and Student 7 exchanged his responses to questions a) and b).

The definition of the level of significance

In general, our students did not consider Items 1 and 2 to be equivalent, although this did not imply a correct interpretation of the significance level. We found a variety in their interpretations: Student 1 considered both Items to be false; Student 5 considered both Items to be true, but not equivalent; Student 4 responded consistently and correctly to both Items, clearly distinguishing the conditioning and conditioned event; Students 2, 3, 6 and 7 considered Item 2 as false and Item 1 as true so that they seemed to exchange the events in the conditional probability defining α. Nevertheless, as we have remarked, all students except Student 7 interpreted the conditional probabilities correctly. In general, for these students the level of significance was associated with the certainty of the null hypothesis and, for some of them, "accepting the alternative hypothesis" was not equivalent to "rejecting the null hypothesis": *"The level of significance is associated with a decision in favor of H_1, the probability of being wrong when accepting H_1, in the case of H_0 being true. For H_0, we should consider error β"* (Student 3). In the same way, for Student 5 Item 2 was true because it referred to the truth of the null hypothesis. This could explain the apparent inconsistency of some students' responses to Items 1 and 2 and the fact that the argument suggested by Falk (1986) and Birnbaum (1982) did not appear explicitly in the analysis of our students' arguments (Vallecillos, 1994).

Confusion between the null and alternative hypotheses

There was some inconsistency between the properties assigned by the students to the null hypothesis, at a theoretical level, and the applications of these properties in the medical diagnosis problem. Although our students considered that the null hypothesis is stated only to be discarded, they chose "having cancer" as the null hypothesis, when we suggested them to interpret this problem from the hypothesis testing perspective. However, in medical diagnosis the hypothesis to be rejected is "to have good health", because, usually, a medical test is not prescribed unless the presence of some illness is suspected. Moreover, in the logic of the test of hypotheses all the computations are done assuming the null hypothesis is true, and in the medical diagnosis problem "to have cancer" is less probable in the whole population than having good health.

Illusion of probabilistic proof by contradiction

All the students, except Student 2, were convinced of the possibility of computing the probability of a hypothesis, so that they could be classified as sharing the "*illusion of the probabilistic proof by contradiction*" (Falk and

Greenbaum, 1995): *"The test establishes that this is true with a given probability ", "It establishes the probability of trueness of the hypothesis"* (Student 1). In some cases, such as Student 1, this belief was reinforced by the interpretation he accorded to the p-value. He conceived the p-value as the probability that the null hypothesis was true (and 1-p as the probability of the alternative hypothesis being true). Other subjects, such as Student 3, suggested a frequential procedure to compute this probability: *"I would take many samples and I would check in how many of them the hypothesis is fulfilled"*. Consequently, the misinterpretation of the level of significance, in our students was due to their confusion between the null and the alternative hypotheses and to their belief in the possibility of computing the 'a posteriori' probability of the hypothesis, given the sample data. It was consistent with their conception of the overall logic of hypothesis testing. There was no confusion in the conditional probabilities P(rejecting $H_0 |$ given H_0 is true) and P(H_0 be true $| H_0$ has been rejected), but the students interpreted the level of significance as this second probability, i.e., the 'a posteriori' probability of the hypothesis, given the data.

CONCLUSIONS

In this paper, we have presented the main results of an experimental study on the interpretations given by university students to the significance level in hypothesis testing through the analysis of interviews. The results of the interviews permitted us to study the different explanatory factors that could induce the misinterpretations of the level of significance, suggested by Falk (1986) and Falk and Greenbaum (1995). In our sample it was not possible to maintain the supposition of confusion in the conditional probabilities defining the level of significance. The verbal ambigüity could not induce the idea of a simple probability referring to only one event or to the conjunction of two events. Our results reinforce the attribution of these difficulties to the *"belief on the probabilistic proof by contradiction"* described by Falk and Greenbaum (1995) as well as to some confusion between the roles of the null and alternative hypotheses. Our students did not consider the test of hypotheses as a decision process, but as a mathematical procedure of (probabilistic) proof of a hypothesis whose trueness or falseness had to be established (Vallecillos, in press).

This lack of understanding is 'reasonable', if we consider the radically different nature of statistical reasoning as compared to the deductive reasoning in other parts of Mathematics. Other reasons are due to semantic difficulties and to the diffusion of the same errors in some text books (Brewer, 1986). Finally, the controversies within the philosophy of science (Rivadulla, 1991; Lehmann, 1993) concerning the possible inductive nature of statistical reasoning and the errors in applying statistical inference in research work are evident signs of the epistemological difficulty of the subject.

All these findings point to the complexity of the meaning of statistical tests. Failing to acquire fundamental aspects of concepts involved in this procedure turns out to be the main factor that could explain the wide misuse and misinterpretation of tests of hypotheses. Consequently, we suggest the need to reinforce the teaching of these topics, which would allow the students to acquire the "elements of the meaning" (Godino and Batanero, 1994) of hypotheses testing, to overcome the biases and errors described and would contribute to the correct application of statistical inference in their future professional work.

Acknowledgments: This research was supported by DGICYT grant PS93-0196, M.E.C. (Spain). We also thank the participant students for their cooperation.

REFERENCES

Birnbaum, I. (1982). Interpreting statistical significance. *Teaching Statistics*, 4, 24-27.

Brewer, J. K. (1986). Behavioral statistics textbooks: source of myths and misconceptions?. In R. Davidson y J. Swifft (Eds.): *Proceedings ICOTS II* (pp. 127-131). University of Victoria.

Diaconis, P. & Freedman, D. (1981). The persistence of cognitive illusions. *The behavioural and brain sciences*, 4, 333-334.

Falk, R. (1986). Misconceptions of statistical significance. *Journal of structural learning*, 9, 83-96.

Falk, R. & Greembaun, C. W. (1995). Significance tests die hard. *Theory &*

Psychology, 5(1), 75-98.

Godino, J. D. & Batanero, C. (1994). Significado institucional y personal de los objetos matemáticos. *Recherches en Didactique des Mathématiques*, 14(3), 325-355.

Lehmann, E. L. (1993). The Neyman-Pearson theories of testing hypotheses: one theory or two?. *Journal of the American Statistical Association*, 88, 1242-1249.

Menon (1993). Statistical testing should be discusses in Mathematics Education Research. *Mathematics Education Research Journal*, 5 (1), 4-18.

Morrison, D. E. Henkel, R. E. (Eds.). (1970). *The Significance Tests Controversy*. Chicago: Aldine.

Rivadulla, A. (1991). *Probabilidad e inferencia científica*. Barcelona: Anthropos.

Vallecillos, A. (1994). *Estudio teórico-experimental de errores y concepciones sobre el contraste estadístico de hipótesis en estudiantes universitarios*. Ph. D. University of Granada.

Vallecillos, A., Batanero, C. & Godino, J. D. (1992). Student's understanding of the significance level on statistical tests. *Proceedings of the XVI PME Conference*, 1, 49-56.

Vallecillos, A. (in press). Conprensión de la lógica del contraste de hipótesis en estudiantes universitarios. *Recherches en Didactique des Mathematiques*

Appendix: Questionnaire

Item 1:

A level of significance of 5% means that, on average, 5 out of every 100 times that we reject the null hypothesis, we shall be wrong (True/False*). Reason your answer:

Item 2:

A level of significance of 5% means that, on average, 5 out of every 100 times that the null hypothesis is true, we reject it (True*/False). Reason your answer.

Medical Diagnosis Problem:

In a hospital, the effectiveness of a diagnostic test to detect cancer has been studied in a sample of 3,000 volunteers. 1,000 of these subjects suffered from cancer and 2,000 were healthy people. In the following table we show the results of the test:

Results of the cancer diagnostic test

Subject group	Positive	Negative	Total
Cancer	900	100	1000
Healthly	20	1980	2000
Total	920	2080	3000

a) Suppose one of these subjects has cancer. What is the probability that his/her test produce a positive result? b) If we take at random one of the positive tests in this sample, what is the probability that the result corresponds to a cancer patient? c) What kind of diagnosis errors might you have in this test? What are their probabilities?

STUDENTS' AWARENESS OF THE DISTRIBUTIVE PROPERTY

Nelis Vermeulen
Cape Technikon, Cape Town

Alwyn Olivier and Piet Human
University of Stellenbosch, Stellenbosch

In this paper we briefly compare the role of the distributive property in arithmetic calculation and in algebraic manipulation. We formulate a model describing different levels of awareness of the distributive property. We describe a teaching experiment to facilitate students' awareness of the distributive property and discuss the outcomes of this experiment.

Introduction

Students' difficulties in learning the basic ideas of early algebra has been well documented. Kieran (1989) emphasises that an important aspect of this difficulty is students' difficulty to recognise and use *structure*. Structure includes the "surface" structure (e.g. that the expression $3(x + 2)$ means that the value of x is added to 2 and the result is then multiplied by 3) and the "systemic" structure (the equivalent forms of an expression according to the properties of operations, e.g. that $3(x + 2)$ can be expressed as $(x + 2) \times 3$ or as $3x + 6$).

Kieran sees algebra as the formulation and manipulation of general statements about numbers, and hence hypothesises that children's prior experience with the structure of numerical expressions in elementary school should have an important effect on their ability to make sense of algebra. Booth expresses the same view:

> ...a major part of students' difficulties in algebra stems precisely from their lack of understanding of arithmetical relations. The ability to work meaningfully in algebra, and thereby handle the notational conventions with ease, requires that students first develop a semantic understanding of arithmetic. (Booth,1989, p. 58)

From this perspective Booth formulates two tasks for research:

- To examine students' recognition and use of structure and how this recognition may develop.
- To devise new learning activities and environments to assist students in this development.

Our study shares the same assumptions about the importance of understanding numerical structure as a prerequisite for understanding algebraic structure. This paper reports on a five year study which addressed Booth's two research tasks, as it relates to the development of students' awareness of the distributive property of operations.

The relationship between arithmetic calculation and algebraic manipulation

The essential nature of any non-counting computational algorithm is that it is a set of rules for breaking down or transforming a calculation into a set of easier calculations of which the person already knows the answers (Olivier 1992). This process of changing the task to an equivalent but easier task involves three distinguishable sub-processes, illustrated here for a procedure to calculate 8×23:

1. *Transformation of the number(s) to more convenient numbers*, e.g. $23 = 20 + 3$.
2. *Transformation of the given computational task to a series of easier tasks*, e.g. the instruction $8 \times (20 + 3)$ is transformed to the equivalent task $8 \times 20 + 8 \times 3$.
   The ability to transform the task to an equivalent task depends on the student's awareness of certain properties of operations, here the distributive property of multiplication over addition.
3. *Computation*, e.g. $8 \times 20 = 160$; $8 \times 3 = 24$; $160 + 24 = 184$.
   These final calculations are performed at the "automatic" level, i.e. without much thinking and requires knowledge of basic number facts.

The process of calculation does not necessarily follow this sequence. The way you initially decompose the numbers depends on your anticipation, "looking ahead" to the transformation you intend in phase 2 and the basic facts you are going to use in the final phase of the calculation. Such a computational strategy therefore includes a realisation of the significance or usefulness of the transformations.

The same task transformation underlies "algebraic manipulation": Finding the product of 8 and $2x + 3$ involves the transformation of the product $8(2x + 3)$ to the sum $8 \times 2x + 8 \times 3$. Human (1988) illustrates this similarity as follows:

Arithmetic			**Algebra**
8×23	$= 8 \times (20 + 3)$	*Transformation of number(s)*	$8(2x + 3)$
	$= 8 \times 20 + 8 \times 3$	*Transformation of task*	$= 8 \times 2x + 8 \times 3$
	$= 160 + 24 = 184$	*Calculation*	

In both the arithmetic and the algebraic case, the transformation from a product to a sum is determined by the distributive property. Explicit awareness of this principle is essential for students when they embark on a first course in elementary algebra. Equally important for these students is understanding that the similarity is confined to the sub-process of task transformation, i.e. that algebra is not "calculation with letters".

Awareness of the distributive property in the elementary school

The traditional standard algorithm for long multiplication involves the same transformations as described above. However, it is probably non-controversial to say that the algorithm has traditionally been taught instrumentally (Skemp, 1989), that the transformations were very much "hidden", and that students in traditional classrooms were not explicitly aware of the logic or the significance of the underlying transformations.

On the other hand, in reform classrooms in South Africa no specific strategies are prescribed and strategies are not "taught". Students are free to construct their own computational strategies. In the process they invent and use a variety of strategies (see e.g. Murray, Olivier & Human, 1994). This is a typical strategy used by young children to calculate 8×23:

$$\begin{aligned} 8 \times 10 &= 80 \\ 8 \times 10 &= 80 \\ 8 \times 3 &= 24 \\ 80 + 80 + 24 &= 184 \end{aligned}$$

Strategies such as these that children in reform classrooms use in solving word problems generally indicate that they have a sound intuitive grasp of the properties of operations that underlie their strategies. Vergnaud (1988) refers to students' intuitive use of properties of operations as "theorems-in-action". Theorems-in-action are physical or mental actions performed by a student, often in a disguised or subtle way, that provide behavioural evidence of implicit knowledge of a more formal property or method or "theorem" of mathematics.

Students' use of theorems-in-action in their calculations led us to hypothesise that students in reform classrooms are explicitly aware of the distributive property. However, this is not substantiated by research. In interviews with grade 5 students (Vermeulen, 1991) only 2 of 16 students (13%) responded positively to the question: "Will $37 \times (13 + 26)$ produce the same answer as $37 \times 13 + 37 \times 26$?". This was later followed by a questionnaire to grade 5 and 6 students which included the following question:

Will the following produce the same answers? Supply a reason for your answer.

1. 37×876 *and* $37 \times 344 + 37 \times 532$
2. 58×356 *and* $58 \times 300 + 58 \times 50 + 58 \times 6$

159 of the 240 students (66%) responded positively to both questions and could state a meaningful reason. We attributed the difference in the results of the interviews and the questionnaire to the use of brackets during the interviews. These results motivated the present teaching experiment to attempt to explicate elementary school students' intuitive knowledge of the distributive property.

Based on our observations and our theoretical framework, we designed the following model for levels of awareness of the distributive property:

Level 1: The spontaneous utilisation of the distributive property: When elementary school students who are free to construct their own computational strategies in arithmetic spontaneously use the distributive property.

Level 2: The recognition of the distributive property: When a student responds positively to questions like "Will 37×52 and $37 \times 30 + 37 \times 22$ produce the same answer?", and can supply a meaningful reason (e.g. 52 is broken down into $30 + 22$).

Level 3: The intentional utilisation of the property: The student intentionally investigates whether utilisation of the property will produce an equivalent number expression which is easier than the original one. Such a student will, for example, replace the expression $27 \times 13 + 27 \times 7$ by the equivalent expression 27×20 in order to simplify the calculation.

Level 4: The generalisation of the property: This entails two different aspects:

- The recognition that the distributive property applies to *all* real numbers, including very large numbers, decimal and common fractions, and (for older students) also to numbers represented by letter symbols.
- The ability to articulate the distributive property.

Level 5: The explanation of the property: Level 5 awareness is demonstrated when a student who has demonstrated level 2 awareness, can supply a meaningful *reason* for questions like: "*Why* do we get the same answer when we multiply both 30 and 22 by 37, as when we multiply 52 by 37?" The distributive property is of course an axiom of the real number system, and therefore cannot be proved. However, students' understanding of the property is based on their earlier concrete experiences. Level 5 awareness will thus be demonstrated if a student can substantiate the distributive property by referring to or connecting it to those earlier experiences.

Designing learning activities to explicate the distributive property

During 1994 we executed a teaching experiment which took the form of developmental research, as described by Gravemeijer (1994). We attempted to design learning activities through which we hoped to explicate grade 6 students' knowledge of the distributive property. By comparing pre-test and post-test results and studying student responses in class, we continuously tried to improve these activities such that they would enable students to explicate their theorems-in-action and lead to a higher level of awareness.

Our frame of reference is very much constructivist. Our intention in designing these activities was therefore that the activities should lead to cognitive conflict and that accommodation should take place. The challenge in designing the activities was evidently how to create such a disequilibrium that would facilitate successful reflection or reflective abstraction by the students (Inhelder, 1974).

It should be stressed that here it is not about forming new cognitive structures (in the Piagetian sense of the word), but rather to make students aware of knowledge which they do possess, but of which they are not explicitly aware. As such it could be argued that an attempt was to be made to transform "unknown" knowledge structures into "known" structures, that is cognitive structures of which students are explicitly aware, and which can be integrated into their existing cognitive structures.

We now briefly describe and discuss some examples of these initial learning activities:

Concrete problems: Students solved concrete problems, such as: "9 children each receive an ice cream of R2.83 and a chocolate of R2.17. What is the total cost?".

Students typically used one of two methods, namely either first adding 2.83 and 2.17 and then multiplying it by 9, or multiplying each of 2.83 and 2.17 by 9 and then adding the products. This represents a concrete manifestation of the distributive property. Through comparing and discussing their different computational strategies, we hoped that it would address their implicit knowledge of the distributive property.

Decontextualised problems: Students were asked to calculate the answer of a problem such as 15 × 36. Many students would break down the 36 into 30 + 6, and then multiply 15 with both 30 and 6, hence using the distributive property. When students were challenged as to *why* he or she does it in such a way, and whether it is *correct* to do it like that, most students gave answers like: "It works out correctly, so surely it is correct?".

In our opinion, based on our observations of student reactions, decontextualised problems, as well as concrete problems, representing slightly different manifestations of the distributive property, did not succeed in bringing learners in touch with their theorems-in-action of this property. We were convinced that is was mainly because in both cases we were not able to create a disequilibrium (in Piaget's sense of the word), thereby not succeeding in forcing students to reflect.

Two equivalent number expressions: "Predict (do not calculate) whether the following two number expressions will produce the same answer. Give a reason for your answer".

25×135 and $25 \times 98 + 25 \times 37$

A variation of this question was to first predict, and then calculate the answer of each expression using a calculator.

To the first type of questions, a number of students readily responded that the answers would be the same, because the 135 was "broken up". When asked why one may "break up" a number, and then multiply with each of the "broken up parts", the answer was once again to the nature of: "It works out correctly, so surely it is correct?".

Intentional application of the distributive property: Here we gave students exercises such as: Calculate without a calculator: $43.6 \times 9 + 43.6 \times 5 + 43.6 \times 6$. Sometimes we would add: "...using the easiest method".

This type of activity did not seem to be successful – even after numerous exercises of the first three types described above, only the minority of students would intentionally first add the 9, 5 and 6, and then multiply the sum by 43.6.

Moving towards a teaching strategy

During 1994, through our observation of students' reactions, and our own reflection, coupled with our constructivist framework of knowledge, we became convinced that a successful teaching strategy should be based on the following principles:

- A powerful cognitive conflict should be created.
- Organised groupwork should be utilised, facilitating better opportunities for students to reflect.
- Calculators should be actively integrated into the learning process, supplying necessary feedback as a source of confirmation or cognitive conflict.

A teaching strategy which accommodated these principles was designed and implemented. We now briefly describe the essential characteristics of the teaching strategy:

- Students solve about three concrete problems. Essentially, there are two approaches to do the calculation, as explained previously. Students are then required to discuss and compare their answers and *methods* with group members, and are led to recognise that there are essentially two methods of calculation.
- Students then calculate the answers of given pairs of equivalent number expressions, using calculators, similar to the example described above. They are required to compare the answers of the two number expressions. For most students it was a surprise to realise that the answers of the two number expressions were equal. They were then challenged to explain *why* the two answers were equal. In our opinion, based on our observation of the students' reactions, this created quite a powerful disequilibrium, and the subsequent effort to try explain this result, could lead to a relatively high level of reflection, especially when students worked in groups.
- To facilitate the process of reflection, students were referred to the previous assignment, that is, the concrete problems. They were challenged to look for and describe similarities between that situation and the current one. Hence, reflection is forced onto a higher level, since they have to reflect on two apparently unconnected manifestations of the distributive property. In this process of reflecting about their reflection of the two separate situations, it is hoped that they will get in touch with their theorems-in-action about the distributive property.
- More problems, stressing the same principle, are then tackled in groups in order to explicate students' awareness even further.

Evaluation of the teaching strategy

During 1995 we implemented the teaching strategy in nine schools, involving 645 students from grades 3 to 7. Students worked together in groups of about four, and each student received a set of work sheets. There were seven work sheets in total, each consisting of four parts.

In evaluating the teaching strategy, we followed a typical pre-test, treatment, post-test design. As described above, students in reform classrooms have a good intuitive grasp of the distributive property at level 1. Given that the objective of the study was to increase students' level of awareness of the distributive property, we designed questions to test level 2 and 3 awareness. We did not test for level 4 and 5 awareness.

There were essentially two types of questions in both tests, determining level 2 and 3 awareness. Examples of the type of question are:

- **Level 2**: *Predict whether the following two number expressions will produce the same answer. Give a reason for your answer.*

  37×150 and $37 \times 95 + 37 \times 55$

  If a student responded positively and was able to supply a meaningful reason, he or she was categorised as displaying level 2 awareness.

- **Level 3**: *Calculate, without using a calculator, the value of the following. It is important to show your work:*

  $18.3 \times 13 + 18.3 \times 7$

  If a student first added 13 and 7, and then multiplied the sum (20) by 18.3, he or she was categorised as displaying level 3 awareness. A student could fall in both categories.

Results and discussion

The results indicated that grade 6 and 7 students experienced a substantial increase in their awareness of the distributive property, as the sample data below illustrates. The teaching strategy was therefore fairly successful.

Results for grade 6 and 7 students*

PRE-TEST				POST-TEST			
level 2 awareness		level 3 awareness		level 2 awareness		level 3 awareness	
N	%	N	%	N	%	N	%
100	54.3	21	11.4	136	73.9	82	44.6

* 184 students wrote the pre-test and the post-test

The outcomes for grade 3 and 4 students were, however, very inconsistent, leading us to conclude that this teaching strategy did not benefit them in acquiring a higher level of awareness of the distributive property. A possible explanation for this is that the younger students found the calculations in the test questions very complex – their thought processes were occupied in trying to manage the calculations and not towards reflection on these thought processes. Using calculators in the test situation may change this.

Also, and this applies equally to the grade 6 and 7 students, we observed during the teaching experiment that the level of classroom discourse (Cobb, 1987) was a critical factor in determining the success of the teaching strategy. It became evident that left on their own, groups seldomly reached the level of discussion necessary for in-depth reflection to occur. It is clear that the teacher has a vital role to play in steering the group discussions in the correct direction.

REFERENCES

Booth, L.R. (1989). A question of structure. In S. Wagner and C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra*. NCTM. Reston, Virginia.

Cobb, P. (1987). An investigation of young children's academic arithmetic contexts. *Educational Studies in Mathematics, 18*, 109 - 124.

Freudenthal, H. (1983). Major problems of mathematics education. In Marilyn Zweng (Ed.), *Proceedings of the Fourth International Congress on Mathematics Education*. Iowa.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education, 25*(5), 443-471.

Human, P. (1988). Problem transformation: a fundamental strategy in arithmetic and algebra. *Pythagoras, 17*, 6-10.

Inhelder, B., Sinclair, H. & Bovet, M. (1974). *Learning and the Development of Cognition*. Harvard University Press. Massachusetts.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner and C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra*. NCTM. Reston, Virginia.

Murray, H., Olivier, A. & Human, P. (1994). Fifth graders' multi-digit multiplication and division strategies after five years' problem-centered learning. In J. P. da Ponte & J. . Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, Vol 3, 399*-406. Lisbon, Portugal.

Olivier, A. (1992). Computation revisited: Nurturing theorems-in-action. In T. Marsh (Ed.), *Proceedings of the Twelfth National Congress on Mathematics Education* (pp. 173-189). The Mathematical Association of South Africa. Port Elizabeth, South Africa

Skemp, R.R. (1989). *Mathematics in the Primary School*. Routledge. London.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds), Number Concepts and Operations in the Middle Grades (pp. 141-161). NCTM. Reston, Virginia

Vermeulen, C. F. (1991). Senior primêre leerlinge se begrip van sekere algemene getaleienskappe, met besondere verwysing na die distributiewe eienskap. [Senior primary student's understanding of properties of operations, in particular the distributive property.] Unpublished M. Ed. dissertation, University of Stellenbosch.

Vermeulen, C. F. (1995). *Verhoging van laerskoolleerlinge se vlak van bewustheid van die distributiewe eienskap in rekenkunde*. [Facilitating primary students' level of awareness of the distributive property in arithmetic.] Unpublished D. Ed. dissertation. University of Stellenbosch.

PRE-SERVICE TEACHERS' CONCEPTIONS AND BELIEFS ABOUT THE ROLE OF REAL-WORLD KNOWLEDGE IN ARITHMETIC WORD PROBLEM SOLVING

Lieven Verschaffel, Erik De Corte and Inge Borghart
Center for Instructional Psychology and Technology
University of Leuven, Belgium

ABSTRACT

Recent research has documented that many pupils show a strong tendency to exclude real-world knowledge from their solutions of school arithmetic word problems. In the present study a test consisting of 14 word problems - half of which were problematic from a realistic point of view - was administered to a large group of students from three different teacher training institutes in Flanders. For each word problem, the student-teachers were first asked to solve the problem themselves, and afterwards to evaluate four different answers given by pupils. The results revealed a strong tendency among student-teachers to exclude real-world knowledge from their own spontaneous solutions of school word problems as well as from their appreciations of the pupil answers.

THEORETICAL BACKGROUND

For several years it has been argued that considerable experience with traditional school arithmetic word problems develops in pupils a strong tendency to exclude real-world knowledge from their problem-solving processes. Rather than functioning as authentic contexts that invite or even force pupils to use their commonsense knowledge and experience about the real world, school arithmetic word problems have become artificial, puzzle-like tasks that are perceived as being separate from the real world. Recent studies by Greer (1993) and Verschaffel, De Corte and Lasure (1994) have yielded strong empirical evidence for this argument. In these studies, a paper-and-pencil test consisting of two kinds of word problems was collectively administered to a group of 11-13-year-old pupils:

- standard problems (S-problem) which can be properly modeled and solved by the straightforward application of one or more arithmetic operations with the given numbers (e.g., "Steve has bought 4 ropes of 2.5 meters each. How many ropes of 0.5 meter can he cut out of these 4 ropes?"), and
- problems in which the mathematical modeling assumptions are problematic (P-problem), at least if one seriously takes into account the realities of the context

called up by the problem statement (e.g., "Steve has bought 4 planks of 2.5 meters each. How many planks of 1 meter can he saw out of these 4 planks?").

The analyses of the pupils' reactions to the P-items yielded an alarmingly small number of realistic responses or comments based on realistic considerations (e.g., responding the above-mentioned "problematic" problem with "8 planks" instead of "10 planks", because in real life one can only saw 2 planks of 1 meter out of a plank of 2.5 meter).

According to the above-mentioned researchers, this tendency among pupils is mainly caused and strengthened by the following two aspects of the current instructional environment: (1) the impoverished and stereotyped diet of standard word problems, which can always be modeled and solved through the straightforward use of one or more arithmetic operations with the given numbers; (2) the way in which these problems are considered and used in the classroom practice and culture, and more specifically the lack of systematic attention to the modeling perspective by the teacher.

In the present study we focused on a major component of the instructional environment, namely the teacher. More specifically, we analyzed (future) teacher's conceptions and beliefs about the role of real-world knowledge concerning the problem context in the modeling of school arithmetic word problems, as reflected by (1) their own spontaneous responses to a set of word problems with problematic modeling assumptions, as well as by (2) their evaluations of pupil answers to these problems that do or do not take into account relevant real-world knowledge.

DESIGN

Participants were 332 pre-service teachers from three teacher training institutes in Flanders. About two thirds were pre-service teachers who had just started their first year of training, while one third were third-year students who had almost completed their pre-service training.

A paper-and-pencil test was constructed consisting of 14 word problems: seven non-problematic standard items (S-items) and seven problematic items (P-items), which are listed in Table 1.

Table 1. The seven P-items involved in the study

---

450 soldiers must be bused to the their training site. Each army bus can hold 36 soldiers. How many buses are needed? (BUSES)

Bruce and Alice go to the same school. Bruce lives at a distance of 17 kilometers from the school and Alice at 8 kilometers. How far do Bruce and Alice live from each other? (SCHOOL)

John's best time to run 100 meters is 17 seconds. How long will it take to travel 1 kilometer? (RUNNER)

This flask is being filled from a tap at a constant rate. If the depth of the water is 4 cm after 10 seconds, how deep will it be after 30 seconds? (This problem was accompanied by a figure of a clearly cone-shaped flask partly filled with water) (FLASK)

A man wants to have a rope long enough to stretch between two poles 12 meters apart, but he has only pieces of rope 1.5 meters long. How many of these pieces would he need to tie together to stretch between the poles? (ROPE)

Steve has bought 4 planks of 2.5 meter each. How many planks of 1 meter can he get out of these planks? (PLANKS)

Carl has 5 friends and Georges has 6 friends. Carl and Georges decide to give a party together. They invite all their friends. All friends are present. How many friends are there at the party? (FRIENDS)

The test was given twice to all pre-service teachers, but each time with a different task. The first time (= Test 1), the student-teachers had to answer the 14 word problems themselves; calculations and comments could be written down in a "comments box" below the "answer box". Immediately after they had finished and handed in this test, they were given Test 2, in which they were asked to score four different answers from pupils to the same 14 word problems as in Test 1 with either 1 point ("absolutely correct answer"), 0 points ("completely incorrect answer") or 1/2 point ("partly correct and partly incorrect answer"). The four response alternatives to the seven P-items in Test 2 belonged to four different categories:

- Non-realistic answer (NA), which results from the straightforward and uncritical application of the arithmetic operation elicited by the problem statement (e.g., for the above-mentioned "planks"-problem, the NA was 10, - the product of 4 times 2.5).
- Realistic answer (RA), which follows from the effective and appropriate use of real-world knowledge about the context elicited by the problem statement (the RA for the "planks"-problem was 8, - the product of 4 times 2).
- Technical error (TE), which results from the straightforward and uncritical application of the arithmetic operation elicited by the problem statement, but which differs from the NA because of a purely technical mistake in the execution of the

arithmetic operation(s). (e.g., responding the "planks"-problem with 100 planks in stead of 10 because of disregarding the decimal number in the multiplication 4 times 2.5).

Other answer (OA), involving an answer that could not be classified into one of the former categories; for instance, solving the "planks"-problem with the result of a wrong operation, such as the addition instead of the multiplication with the two given numbers 4 and 2.5.

At the bottom of each problem, there was a box for writing explanations or comments.

For Test 1, an assessment of the student-teachers' spontaneous solutions to the P-items was made involving two major categories: realistic reaction (RR) versus non-realistic reaction (NR). The term "realistic reaction" (RR) refers to each case wherein a student-teacher gave either a realistic answer (RA), or an answer that was scored differently (NA, TE or OA) but that was accompanied by a realistic consideration in the comments box. A non-realistic reaction (NR) refers to each case wherein no evidence of activation of real-world knowledge could be found in the answer box nor in the comments box. The analysis of the pre-service teachers' reactions to the seven P-items in Test 2 focused on the score (1, 1/2 or 0) given to the realistic answer (RA) and the non-realistic answer (NA).

RESULTS

Test 1. As expected, the student-teachers demonstrated a strong overall tendency to exclude real-world knowledge and realistic considerations when confronted with the problematic word problems. Indeed, only 48 % of all reactions to the seven P-items from Test 1 could be considered as realistic (RRs). This percentage is considerably higher than in previous studies with upper elementary and lower secondary school pupils (Greer, 1993; Verschaffel et al., 1994), where overall percentages of RRs between 15 % and 20 % were observed. Nevertheless, it remains disappointingly low, as it implies that in more than half of the cases, the student-teachers solved the P-items from Test 1 in a stereotyped, uncritical way, without any consideration for the realities of the context involved in the problem.

There was a significant difference in the overall number of RRs between the first-year and the third-year student-teachers in favor of the latter group (t-test, two-tailed, $t = 3.40$, $p < .01$). However, the overall percentage of RRs remained low in both groups, namely 45 % and 54 % for the first-year and the third-year students, respectively. Interestingly, additional t-tests revealed that this difference between both years was significant in only two of the three teacher training institutes involved in the study. This suggests that student-teachers' disposition toward realistic modeling of arithmetic word problems is at least partially influenced by the courses on mathematics education received during their pre-service training.

Test 2. The student-teachers' strong disposition toward non-realistic modeling was also revealed by their evaluations of the realistic answer (RA) and the non-realistic answer (NA) on the same seven P-items during Test 2. Only in 47 % of the cases the RA received a score of 1; 6 % of the RAs received a 1/2-score and in 47 % of the cases the RA was scored with a 0. On the other hand, the NA was scored with a 1 and a 1/2 in 56 % and 26 % of the cases, respectively; only 18 % of the NAs received a 0-score. Thus the student-teachers' overall evaluation of the stereotyped, non-realistic answer to the P-items was considerably more positive than for the realistic answer based on context-based conderations.

There was again a significant difference between the first-year and the third-year student-teachers. The third-year students gave significantly more 1-scores (t-test, two-tailed, $t = 3.29$, $p < .01$) and less 0-scores (t-test, two-tailed, $t = 2.63$, $p < .01$) for the RAs than the first-year students. Correspondingly, the third-year students produced significantly less 1-scores (t-test, two-tailed, $t = 2.30$, $p < .05$) and more 0-scores (t-test, $t = 2.33$, $p < .05$) for the NAs than the first-year students. Additional t-tests revealed that these four differences between first- and third-year students were significant in only one of the three teacher training institutes.

Relationship between Test 1 and Test 2. The previous sections focused on the results for Test 1 and Test 2 separately. In this section, we will investigate to what extent the student-teachers' evaluations of the NAs and the RAs during Test 2 matched their own performances during Test 1, by analyzing the scores they gave on Test 2 for the RA and the NA following the 52 % non-realistic reactions (NRs) and the 48 % realistic reactions (RRs) on Test 1 separately (see Table 2).

The left part of Table 2 presents the distribution of the different combinations of RA scorings (1, 1/2 or 0) and NA scorings (1, 1/2 or 0) over the seven P-items of Test 2 for the 52 % non-realistic reactions on Test 1, as well as the distribution of the scorings for the RA and the NA over the three scores (1, 1/2 and 0).

As expected, we found a strong relationship between the non-realistic reactions on a P-item during Test 1, and the evaluations of the RA and the NA on that item during Test 2. In 89.3 % of all cases wherein a NR was given to a P-item during Test 1, the NA to that item was given a 1-score in Test 2. Correspondingly, 83.1 % of the NRs during Test 1 were followed by a 0-score for the RA in Test 2. The match of a 1-score for the NA and a 0-score for the RA occurred in no less than 79.3 % of all cases wherein a P-item from Test 1 was answered with a NR. Apparently, the NA was scored with 1 because this response corresponded to the stereotyped, non-realistic answer the student-teachers had given themselves on this item during Test 1, and they scored the RA with 0 because they could not understand, and, therefore, appreciate the context-based considerations underlying this answer.

Ten percent of the NRs to a P-item during Test 1 was followed by a 1-score for the RA during Test 2. This suggests that in those cases the confrontation with the RA during Test 2 had functioned as a scaffold toward (more) realistic modeling. However, the finding that only 10.0 % of the scorings following a NR yielded evidence for the scaffolding effect of the confrontation with the RA, can be interpreted as additional evidence for the strength and the resistance of the tendency toward non-realistic mathematical modeling among student-teachers. Interestingly, these 10 % scorings showing insight into the appropriateness of the RA during Test 2 - as evidenced by the 1-score for the realistic response alternative - were accompanied by a diversity of appreciations of the NA (0, 1/2 and 1). An explanation for these scoring patterns will be given below.

Table 2. Combinations of RA scorings (1, 1/2 or 0) and NA scorings (1, 1/2 or 0) (in percentages) over the seven P-items of Test 2 for the total number of non-realistic reactions (52 %) and of realistic reactions (48 %) on Test 1

NON-REALISTIC REACTIONS (NR)

	RA			
NA	1	1/2	0	Total
1	3.4	6.6	79.3	89.3
1/2	3.8	0.1	2.6	6.5
0	2.8	0.2	1.2	4.2
Total	10.0	6.9	83.1	100.0

REALISTIC REACTIONS (RR)

	RA			
NA	1	1/2	0	Total
1	10.2	3.2	7.4	20.8
1/2	42.3	2.0	1.4	45.7
0	32.9	0.4	0.2	33.5
Total	85.4	5.6	9.0	100.0

The right part of Table 2 presents the distribution of the different combinations of RA scorings (1, 1/2 or 0) and NA scorings (1, 1/2 or 0) over the seven P-items of Test 2 following the 48 % RRs on Test 1. As shown in this part of the table, the congruence between the RRs on Test 1 and the scorings of the RA and the NA during Test 2 was less straightforward than for the NRs. On the one hand, the evaluations of the RA were generally in line with the reactions on Test 1: indeed, 85.4 % of the RRs on Test 1 were followed by a 1-score for the RA on Test 2.

But the scorings for the NA were rather surprising: only 33.5 % of the subjects who reacted in a realistic way to a P-item during Test 1 scored the NA with a 0 during Test 2 (almost always in combination with a 1 for the RA). A closer look at Table 2 reveals that one scoring combination occurred even more frequently than the combination of a 1 for the RA with a 0 for the NA: in 42.3 % of the cases wherein a RR was given to a P-item from Test 1, the 1 score for the RA on that item was accompanied by a 1/2 for the NA. In addition, the combination "1 for RA and 1 for NA" also occurred in a substantial number of cases (10.2 %). These results indicate that in many cases where student-teachers reacted themselves to a P-item in a realistic manner, they were nevertheless quite understanding and tolerant to elementary school pupils who interpreted and solved these P-items without seriously taking into account the relevant knowledge about the context called up by the problem statement. According to their written explanations in the comments box, they thought that it is unfair to punish a fifth-grader for solving the P-item in a stereotyped, non-realistic manner. This is illustrated by the following comment accompanying the scoring combination "1 for RA and 1 for NA" with respect to the runner-item: "I scored alternative D (= the RA: "It is impossible to answer precisely what John's nest time on 1 kilometer will be") with 1 because the pupil who gave this answer knows that is not realistic to assume that John will be able to run at his record speed for 1 kilometer. But I also gave 1 for alternative A (= the NA: "17 x 10 = 170. John's best time to run 1 kilometer is 170 seconds") because from a purely computational point of view this is the correct answer."

Interestingly, a considerable percentage of scoring combinations following a RR during Test 1 involved a score of 1/2 (5.6 %) or even a 0 (9 %) for the RA. A qualitative analysis of the written protocols accompanying these unexpectedly low scores for the RA (taking into account that the student-teacher had produced a RR on this item during Test 1), revealed that the RA was appreciated so moderately because of its vagueness; to deserve a better score, the RA should have been more precisely formulated and/or better motivated. This is illustrated in the following exemplarily comment for a 1/2-score for the RA on the rope-item ("It is impossible to know how many pieces of rope you will need"): "In fact the boy who has answered in this way is right because you do not know how much you will loose for making the ties, but he should have explained this in his answer".

CONCLUSIONS

Recent studies have provided ample evidence for a phenomenon whereby children solving word problems in school often produce answers without regard for realistic constraints (Greer, 1993; Verschaffel et al., 1994, in press). The present study provides some insight into one of the instructional factors that are considered responsible for the development of this tendency among children, namely the teachers' own conceptions and beliefs about the importance of real-world knowledge in arithmetic word problem solving. While this study convincingly demonstrates that many future teachers have knowledge and beliefs about teaching and learning arithmetic word problems that are problematical from a realistic point of view, it does, of course, not yield direct evidence that these teacher conceptions and beliefs are responsible for children's strong tendency to exclude real-world knowledge from their problem solving endavours. However, based on the recent literature on mathematics learning and teaching (De Corte, Greer & Verschaffel, in press), there is good reason to assume that these teacher cognitions and beliefs about the role of real-world knowledge in the interpretation and solution of school arithmetic word problems, have indeed a strong impact on their actual teaching behavior and - consequently - on their students' learning processes and outcomes. Therefore, if we want to connect problem solving in school mathematics to the experiental world of children - as strongly advocated in most current reform documents related to mathematics education - we will also have to stimulate and help (student-)teachers to construct the proper concepts, skills and beliefs that are needed for realistic modeling of problem situations and for realistic interpreting of outcomes of arithmetic calculations, as part of a genuine mathematical disposition.

REFERENCES

De Corte, E., Greer, B., & Verschaffel, L. (in press). Mathematics teaching and learning. In D. Berliner & R. Calfee (Eds.), Handbook of educational psychology. New York: Macmillan.

Greer, B. (1993). The mathematical modeling perspective on wor(l)d problems. Journal of Mathematical Behavior, 12, 239-250.

Verschaffel, L., De Corte, E., & Lasure, S. (1994). Realistic considerations in mathematical modeling of school arithmetic word problems. Learning and Instruction, 4, 273-294.

Semiotics as a descriptive framework in mathematical domains.

Adam Vile and Stephen Lerman

South Bank University.

This paper reports on ongoing research into the application of a semiotic perspective to the analysis of meaning-making in mathematics learning. In this paper we present a version of semiotics, termed developmental semiotics, that augments a Piercian semiotic framework with Vygotskian notions of the action of sign and its role in development. We will suggest that developmental semiotics may be of use for the description of meaning-making in the domain of mathematics learning. We will present developmental semiotics first as a point of view and then as a toolkit which may inform analysis of learning in mathematics education. We will give some brief examples of the use of this toolkit, arising from a case study in algebra, and discuss some of the implications that such a perspective may have for mathematics educators.

Introduction

Semiotics, "the doctrine of signs" (Deely 1991) appears throughout the literature in a number of guises. It has two roots, one in structuralism through the work of Saussure and Barthes and the other in the work of C.S. Pierce, (who coined the term). Each offers a description of the role of the sign in communication and meaning-making yet they differ in two crucial aspects. The first difference being in the action of semiosis, the second in the nature of the analysis. Space does not allow a full exposition of these differences here, suffice it to say that by virtue of its diachronic analysis and the tripartite action of semiosis we have adopted the Piercian perspective, that is the perspective presented by Pierce (in Buchler 1955) Eco (1979) and Deely (1991) amongst others, for this work. This form of semiotics has been used for philosophical analysis (Deely (op. cit.)) literary criticism (Harris 1992) and to a limited extent to inform about educational contexts (Groisman et al. (1991) in science education and Stage (1991) in mathematics education) and mathematics (Rotman 1991).

Quite apart from the semiotic literature arising from philosophical inquiry, semiotics has been applied in cultural psychology to describe the role of language and tools in the mediation of psychological functioning, most notably in the work of Vygotsky (1977, 1978), an application of Vygotsky's semiotic analysis was presented at PME (Ohtani 1994). Vygotsky's semiotics arose quite independently of either the work of Pierce or Saussure and is central to his notion of development towards higher mental functioning and scientific concepts. His description of the action of semiotics closely resembles that of Pierce, and objectifies the diachronicity of the Peircian method into notions of development. In developing a tool kit for semiotic analysis we will draw on many of Vygotsky's notions.

Further connections may be drawn between the work of Pierce and that of Vygotsky when the subject matter of semiotics is considered. Semiotics is about the action of signs, and signs are ever-present, actually and virtually, within our experience and culture:

> signs perfuse all.... the human being structures and applies those signs, and....the semiotic that results in their application....is inextricably bound up with context and culture (Harris 1992 p.8)

Every act of communication is an act of semiosis and the sociocultural view of knowledge inherent in a Vygotskian perspective is based upon the notion that meaning-making occurs as a result of semiotic action. Vygotsky sees semiotics as an act of mediation from the social to the individual through psychological tools (Vygotsky 1977) and in particular through the word- the cultural unit. Pierce sees semiotics as a similar tri-partite action which transcends the subject object divide by way of the interpretant. For Vygotsky the word, for Pierce the interpretant (a sign) makes meaning.

In this we paper present a synthesis of the semiotic notions of Peirce and Vygotsky in an attempt to provide a semiotic perspective on meaning-making in mathematics learning. We will outline the theoretical perspective and then, through the use of examples taken from a case study, will show how the developmental semiotic framework may be used to inform meaning making in the domain of algebra. We will end with some implications of this research.

The developmental semiotic perspective.

Central to a semiotic point of view is the triadic nature of semiotic action. A variety of semiotic triangles exist (for example see Eco 1979) specifying the connection between subject and object through a mediating entity. This entity Pierce termed the *interpretant* which he defined in the following way

> A Sign, or representamen, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. (Peirce in Buchler (1955) p.99)

The interpretant is a dynamic sign which represents the meaning made by an individual in respect of her network of experiences into which a given act of semiosis is inserted. The interpretant is a meaning laden indicator not only of the meaning held by the interpreter but also of the experiences which led to the making of that meaning. We want to extend this notion even further to ascribe prime place to the interpretant not only as the mediator of meaning but of the space in which both individual and social meanings are made, in other words the space at which the intra *and* inter subjectivities meet and in which, as a result, meaning is made.

For example one could take the making of meaning for the equals sign. Barrody and Ginsburg (1982) have shown that at least two meanings may exist, operational or relational. Semiotics describes the meaning-making process in terms of the past experiences of the interpreter and the triadic action of the sign. The sign <=> will be recognised by the interpreter, probably not iconically (nothing is more equal than two parallel lines) but more than likely as an opaque symbol which they have used (experienced) in certain mathematical practices. It may stand to them as a process to carry out (operational) as a statement of equivalence (relational) or both. Whatever, they will make a meaning for that sign as a result of experiences of mathematical practice and of course the current practice in which they are engaged.

Central to this process is the insertion of the act of semiosis into a *network of experiences*; in order to address the question concerning the creation of this network of experiences, which could be classed as the content of the intrasubjective, one must consider semiosis to act diachronically, and it is at this point that the notion of development is brought into the frame. Lev Vygotsky (1977,1978) presents a cultural psychology with semiosis, activity and development as central to the genesis of higher mental functions. In fact his notion of "scientific" concepts (which act in higher mental functioning), as concepts that make meaning through links with other concepts, can be equated with the notion of interpretant, which first acts in semiosis and then becomes a sign. We want to suggest that the network of experiences is a web of interconnected interpretants (acting as signs) that have been internalised diachronically through acts of semiosis.

Developmental, semiotically mediated meaning has been examined by Becker and Varelas (1992) from within a Vygotskian framework. Their study of counting examines the different degrees of semiotic demand required for various activities. They introduce the notion of foregrounding and backgrounding in which the empirical sign-object referent is pushed further to the back as development progresses, thus strengthening the links in the sign-sign plane. The authors admit that their study concentrates on the semiotic aspect of development rather than the conceptual but they also suggest that there may come a time when semiotic and conceptual issues may be difficult to pursue distinctly. Semiotics does not use the term 'concept'; rather we find Pierce talking of *ideas:*

> It [the sign] stands for [an] object, not in all respects but in reference to an idea........ "Idea" is here to be understood .....in the sense in which we say that one man catches another mans idea. (Pierce in Buchler (1995) p. 99)

When one person catches another's *idea* the two are obviously engaged in communication, furthermore they will "catch each other's *ideas*" because of a belief in a shared understanding. Adopting a Wittgensteinian (1974) stance one may extend this notion to consider ideas as central to and arising from social practice. We want to suggest that from this perspective *ideas* (concepts, meanings) are not separable either from semiosis or social practice.

Building on the work of Becker and Varelas, and from within a semiotic position, we may now present a description of meaning-making that attempts to address the issue of the amalgamation of concept and semiotics, and that may provide a description of the meaning-making process that will account for the diachronic nature of experience. We have termed this perspective developmental semiotics.

The essential elements of developmental semiotics are the nature of the interpretant, sign-sign foregrounding, and the internalisation of a network of experiences en route to higher mental functioning. Such a perspective presents development as a continuous process of meaning-making, beginning with making meanings at a concrete sign-object level and progressing towards an unlimited semiosis where meanings are made through sign-sign interaction, each sign having the possibility of then becoming the object in another semiotic triad. Through the action of the interpretant the intersubjective gives way to the intrasubjective and individual higher mental functioning arises as a result of that interpretant becoming a sign and of the relationships that sign has with other intrasubjective signs.

On the macro level developmental semiotics provides a description of meaning. On the micro level it may provide a toolkit for the analysis of specific meanings in specific situations. The interpretant, the space in which intra- and inter-subjectivities meet, is the one variable in semiotic action and as such may provide clues to meanings made in a given action. The notion of development towards sign-sign foregrounding represents a developmental shift towards higher mental functioning and in a specific context interpretants which represent various stages in this shift may be evident. Semiotics provides a vocabulary, and a theoretical framework which will enable discussion about the nature and the role of the sign in the meaning-making process, and the implication is that there be a shift in emphasis to the role of the sign in the analysis of meaning-making in a given mathematical context.

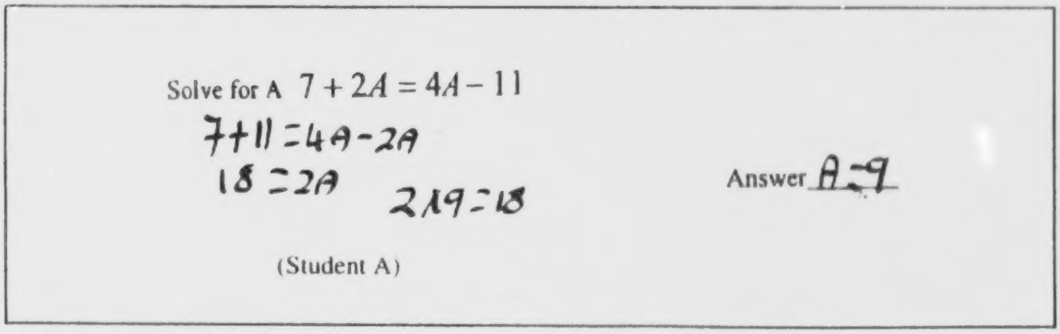
As an elaboration of the developmental semiotic description consider a possible line of development (in a single context) leading to the solution of the linear equation $3x+2=x-4$. In order to solve such an equation a student must be at a certain level in her semiotic development (particularly as this equation is considered to be at the 'other' side of the cognitive gap (Filloy and Rojano 1989)).

At the very start of development she began counting in groups of objects 11, 111, 1111 etc. concretely, with a high degree of sign-object reference (as the sign

<1> represents "one object" iconically). To make a meaning for the symbols <2> and <3> she would have had to background the concrete reference somewhat and associate the abstracted sign <2> with her previous action of grouping 1's, particularly when it comes to make a meaning for addition and then multiplication. In order to build a meaning for <+> she must not only perceive it as the order to count on, or count all she must *know how* to use it intelligently, if she wants to background the more concrete count all procedure (which she must do if she wants to be able to multiply large numbers meaningfully) she must make a more complicated meaning, and begin to use the sign <2> as an entity in its own right. More than *knowing how* she must have built up a network experiences to enable her to *know that* <2> is an entity in its own right.

The same is true of her later development and when she comes to meet the sign <x> she may make a meaning for it as a generalised number, perhaps through experiences of < [ ]+5=7>; alternatively she may make meaning for it as an operation to be carried out, or even (depending on the metaphor) of a weight on a scale. Whatever meaning she makes will be based upon her experiences of the use of the signs <+> <=> etc. and of arithmetic, as well as the degree of sign-sign foregrounding, but more, it must be sufficient enough to enable her to make a meaning for <x> as an entity to be manipulated meaningfully. Thus we see the sign as an opaque compounded entity for which "senseful" (Lins 1994) symbolic meaning must be made.

Below we present some extracts of students solutions to the equation 7+2A=4A-11. This is an equation with a high degree of semiotic demand as solution requires the manipulation of the unknown as well as manipulation of numbers. Semiotically this context is complicated by the number of signs requiring interpretation. Apart from the equation itself, the unknown, operators and numerals are signs for which meanings must be made. In the following brief discussion we will concentrate on the interpretants for the equation and the unknown.



Solve for A $7 + 2A = 4A - 11$

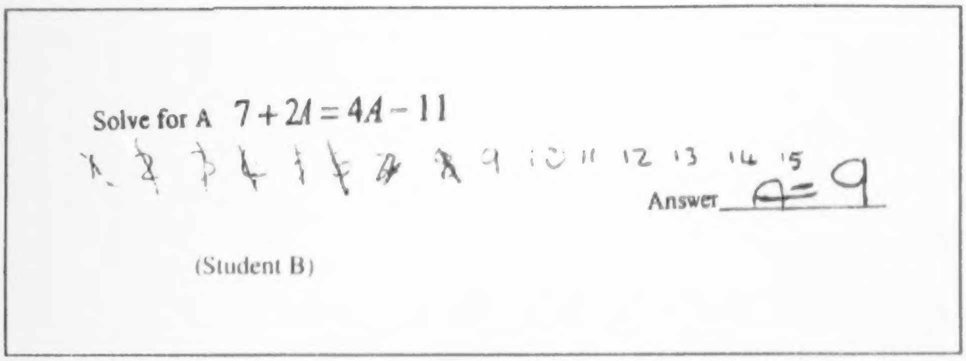
$7+11=4A-2A$

$18=2A$ $\quad 2 \times 9=18$

Answer $A=9$

(Student A)

It appears that student A is using a syntactic metaphor for solution - "move the 2A and -11 to the other side of the equation and change the sign". This indicates a high level of sign-sign foregrounding for both the sign <A> and the sign <7+2A=4A-11>.



Solve for A $7 + 2A = 4A - 11$

~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ ~~6~~ ~~7~~ ~~8~~ 9 10 11 12 13 14 15

Answer A = 9

(Student B)

Student B attempts the same equation with a much lower degree of sign-sign foregrounding but with still a senseful meaning. The extract indicates a use of a trial and error method, <x> is seen as standing for a number and as a sign it does not stand to the student as an object itself merely as the representation for <9> (another sign).

Although this trial and error meaning for the sign <7+2A=4A-11> is senseful, this meaning will not provide the required flexibility to tackle equations with non integer solutions (only integer solutions have been tried). Student A demonstrates flexibility in the ability to make a meaning for the sign <equation> at two levels, depending on the degree of semiotic demand. The equation <18=2A> is solved by recourse to generalised arithmetic and the students network of experiences in the process of doubling- "two times what makes 18?".

These two brief examples demonstrate two extremes of semiotic development. Both extracts indicate that the students have made senseful meanings for the signs through the interpretant in relation to a network of experiences, and the extract from student A suggests further the developmental nature of those meanings. For student B the shift to a more symbolic meaning would be essential for progression.

Implications

Developmental semiotics may offer contributions at two levels. First, at the macro level, it provides support for a sociocultural view of mathematics and situates the meaning making process firmly in communication by making the sign central to its action. Second at the micro level it may offer a toolkit and vocabulary which may prove fruitful in informing the meaning making process in specific domains.

The notion of development is central to the description presented here and, accepting that development as prescription is problematic (Burman 1994), is an integral part of the meaning-making process. Within this perspective it is essential that for progression towards higher mental functioning, signs become compounded into other signs and that senseful meanings be made for each sign. Further work in applying a semiotic perspective to mathematical domains will need to investigate both the nature of senseful meanings and the types of semiotic activity that will help generate such senseful meanings.

Research carried out from within a developmental semiotic perspective will concentrate on the nature of the sign in specific acts of semiosis. This focuses the need for a research methodology aimed at eliciting meanings made in those contexts. Central to the empirical aspect of the ongoing research programme into the application of this perspective to specific domains is the evaluation of unstructured interview and mathematical writing as methods of data capture suitable for providing data for semiotic analysis.

We have suggested that developmental semiotics can inform about meaning making in specific domains and have given examples of arithmetic and algebra as possible domains. The mathematics register consists of complex signs with a high degree of semiotic demand and all mathematical domains may be thought of as sign systems. Investigations into the nature of signs in any mathematical domain should, in theory, be possible and further work would investigate some of these domains from a semiotic perspective, to bring out specific implications for teaching and learning within these domains.

References:

Baroody, A. and Ginsburg, H. (1982). The Effect of Instruction on Children's Understanding of the Equals Sign. Paper presented at the American Educational Research Association, New York. march 1982.

Becker J. and Varelas M. (1992). Semiotic aspects of cognitive development: illustrations from early mathematical development. Psychological Review, 100, 3, 420-431

Buchler, J.(1955). Philosophical Writings of Pierce. Dover publications.

Burman E.(1994) Deconstructing Developmental Psychology. Routlage.

Deely, J. (1991). The Basics of Semiotics. Indiana University Press.

Eco, U. (1979). A Theory of Semiotics. Indiana University Press.

Filloy, E. and Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. For the Learning of Mathematics, 9, 2, 19-25.

Groisman, A., Shapiro, B. and Willinski, J. (1991). The potential of semiotics to inform understanding of events in science education. International Journal of Science Education, 13, 3, 217-226

Harris, A. (1992). From Codpieces to Spoonerisms: Semiolinguistic Approaches to Communication in Culture. Paper presented at the 63rd annual congress of the Western States Communication Organisation.

Lins, R. (1994) Eliciting the meanings for algebra produced by students: knowledge, justification and semantic fields. Proceedings of PME XVIII, Lisbon, 3, 184-191.

Ohtani, M. (1994). Sociocultural mediatness of mathematical activity: analysis of voices in seventh grade mathematics classroom. Proceedings of PME XVIII, Lisbon, 3, 384-391.

Stage, F. (1991). Semiotics In The College Mathematics Classroom. Paper presented at the annual meeting of the American Educational Research Organisation ,Chicago.

Vygotsky, L. (1977). Thought and Language. M.I.T. press.

Vygotsky, L. (1978). Mind in Society. Harvard University Press.

Wittgenstein, L. (1974). Philosophical Grammar. Oxford, UK: Basil Blackwell.

Some Psychological Aspects of Professional Lives of Secondary Mathematics Teachers-
-The humiliation, The frustration, The hope

Shlomo Vinner
The Hebrew University of Jerusalem

ABSTRACT

Some teacher group discussions about problems in their professional life are described and analyzed. It is claimed that these teachers do not show a convincing reflective ability on one hand, and on the other hand, they are both frustrated and humiliated by the system.

Introduction and a methodological comment

A lot of research has been done about teachers. Usually, the studies focus on one particular aspect of their professional life. It might be their mathematical knowledge (for instance, Even, 1990), their educational philosophy (Boufi,1994), their desire to use computers in teaching (Bottino&Furingheti, 1994), their tendency to have problem solving in their everyday practice (Fernandes, 1994) and so on. There are fewer studies, however, which relate to more general aspects of mathematics teachers' professional life. One of them is Shuard&Quadling (1980). Its title even includes part of this paper title: "some aspects of professional life." It opened a gate to this extremely important domain. Some teachers were invited to write reactions to some questions posed to them by the editors of the book. The written reactions were presented to the reader with some editorial comments and introductions. The approach of this study is similar in a way and totally different in another way. Questions were posed to teachers about their professional life, but this was done in group meetings and the teachers, after handing in their short written replies, had long discussions on the issues raised in these short written replies. Very often the discussions went beyond these issues to other important issues of the teachers' professional life and thus reactions to questions which were not originally posed to the teachers were obtained as well. The group meetings were considered as **a framework in which several teachers discuss and share personal problems related to their profession**. This is a modification of the Webster's Ninth New Collegiate Dictionary description of a group therapy and the approach is also similar. Namely: 1. Discuss your own problems and this will make you more aware of them. 2. Listen to other people's problems. 3. Listen to other people's reactions to your problems. All these are supposed to help you solving your problems. The group discussions were videotaped and transcribed. They were watched and read several times. This was a basis for an analysis which led to two main outcomes. The first one is a set of excerpts (part of which will be presented later on). The second is the interpretation that I suggest to the excerpts. As to the excerpts, one could say that they speak for themselves. This is partly true. On the other hand one has to remember that selecting these particular excerpts is already an interpretation of the data. But even after selecting the excerpts, they can still speak in more than on voice. Therefore an interpretation should be added. Of course, there is a methodological problem involved here. This problem cannot be easily solved. It is the methodological problem associated with many case studies and interview analysis. Interpretations are accepted if they are convincing, but it is impossible to give criteria for being convincing. Different people react differently to the same data and interpretation and there is no way to overcome it.

The official name for the group discussions mentioned above was a workshop for discussing problems in mathematics education. And if problems are mentioned also problem solving should be mentioned. Schoenfeld (1985) speaks about four factors involved in problem solving. Although he speaks about mathematical problem solving it seems to me that his conceptual framework is suitable, *mutatis mutandis*, also for other problem domains. This includes also the problems mathematics teachers have in their professional life. Therefore if we want to analyze teachers discussions about their problems Schoenfeld's conceptual framework can be used. The four factors which constitute Schoenfeld's conceptual framework are: 1. Resources. 2. Heuristics. 3. Control. 4. Belief System. A general advice which is given to the problem solver (and this goes back to Polya, 1957): 1. Identify the problem. 2. Devise a plan. When examining the excerpts we can ask ourselves whether the teachers follow these guidelines, whether they identify the problem correctly, what are their resources, what are their belief systems and do they have a control mechanism? Because of space problems I won't be able to get into a systematic discussion of all these. I will just refer to them here and there.

Sample and questionnaire

The entire sample on which this study has been done included several teacher groups who met and interacted in a similar way. In this study I will discuss only one of them (N=11). An additional group, in the context of other issues, is discussed in Vinner (1995). The range of the teaching experience in this group was between 5 and 30 years with a mean of 15. It included 8 females. The following questions were presented to this group in the beginning of a session. The teachers were asked to answer these questions shortly in writing. Later on their answers served as a trigger for a long group discussion.

Questions

1. Note 3 to 4 difficult topics or concepts in the mathematics curriculum. 2. What makes them difficult? 3. How do you cope with these difficulties?

Results and analysis

1. Frustration

Here is the distribution of answers according to the topics or concepts which were mentioned in the written answers. The numbers in parenthesis indicate the number of teachers who mentioned the topic (or concept) in their list

A) The topic: Proofs in Geometry (7)

The cause of difficulty: 1. You cannot find two proof assignments which look exactly the same. 2. Proofs need creative thinking. 3. Proofs require accumulated knowledge. 4. The geometrical tasks require a lot of concentration. 5. The material is abstract.

The way to cope: 1. A lot of class and homework assignment. 2. Pictures and drawings in colors. 3. I give up proofs and their accurate formulation and focus on computational assignment.

B) The topic: Word problems (4)

The cause of difficulty: 1. Reading comprehension problem. 2. Difficulties in translation from everyday language to the language of Algebra. 3. The problems are taken from different domains, contrary to the situation of the technical exercises. 4. The cause of difficulty is not known.

The way to cope: 1. Reading the word problem with the students while trying to understand the mathematical relations underlying there. 2. Explaining again and again in every possible way and giving more assignments. 3. Working gradually.

C) The topic: Trigonometric equations (4)

The cause of difficulty: 1. Too many concepts. 2. The difference between this topic and the concepts which were studied in the past. 3. The need to know trigonometric identities and to use them in the solution process.

The way to cope: 1. Explaining again and again. 2. A lot of exercises.

D) The topic: Derivative (4)

The cause of difficulty: 1. The fear of a new topic. 2. The complexity of the concept which is based on the concept of limit or on the concept of approximation. 3. The concept presents a new form of thought which is unfamiliar to students who are used to think that mathematics is 2+2 = 4.

The way to cope: Frequent repetitions on the concept with the hope that the infinite repetitions will do the job. 3. Pictures and drawings in colors.

E) The topic: Three dimensional geometry (2)

The cause of difficulty: 1. Lack of spatial visualization. 2. Lack of concrete material teaching aids.

The way to cope: 1. Suitable drawings and demonstrations in the classroom space.

F) The topic: Limit (1)

The cause of difficulty: The superficial approach to the concept and the avoidance of a deep discussion of it.

The way to cope: 1. Avoidance of dealing with the concept.

More Topics: Radians (1); Problem with parameters (1); Vectors (1); Graphical solutions of inequalities (1); Inequalities with absolute values (1); Exponentials (1); Logarithms (1); Functions (1); Proofs by induction (1).

The cause of difficulty: Lack of algebraic skills (Graphical solutions of inequalities); Forgetting (exponentials); Lack of internalization (Function); The passage from n = k to n = k+1 (Proofs by induction)

The way to cope: Skipping it (Vectors; Logarithms; Inequalities with absolute values); Start teaching the concept already at the junior high level (Functions); Giving numerical examples (Proofs by induction)

The surprising fact about the above data is its diversity and the number of marginal topics which are not expected to appear on a short list of difficult topics. Here, the reflective ability of these teachers is questioned (This is part of the control, in Schoenfeld's terminology) Teachers who reflect about their teaching would have a short list of central topics which are difficult for their students. Although the word "central" was not mentioned in my question it was expected that, since the number of topics was limited to 3 or 4, concepts like radian or inequalities with absolute values will not be mentioned. (There is no doubt that these are difficult topics.) On the other hand, it would be expected that concepts like functions, limits, or vectors will be considered as difficult topics by more than one teacher. It would be expected that proofs in general will be mentioned and not only proofs in geometry. It would not be expected that the number of answers mentioning word problems will be the same as the number of answers mentioning trigonometric equations (a difficult topic, but not a central one in the curriculum) When examining the causes of the difficulties, the teachers' resources (in Schoenfeld's terminology) should be questioned. At the side of some good answers (although formulated sometimes in a naive language like " the concept presents a new form of thought which is unfamiliar to students who are used to think that mathematics is 2+2=4") one can find useless banal answers. They are useless because they do not give any clue which is particular to the concept in consideration ("the fear of a new topic"). There is no reference, whatsoever, to any learning or cognitive theory. Either the teachers have not been exposed to them or they think they are irrelevant to their everyday practice. Unfortunately, too often, the only question which the everyday practice raises is how to survive. This is expressed very clearly in Shuard and Quadling (1980, p.18) when a new teacher says:

After a good "Team talk" the advice we were given seemed to make more sense than all the first term of Piaget. Instead of all the talk about helping, guiding, and caring, we were given a lecture on "us" and "them" and how to survive...

While the teachers were busy with writing answers to my questions, the following short dialogue took place.

Yacov: *You are upsetting us?* I: *Why?* Yacov: *Because if it is difficult for the students it implies that I do not explain it adequately*

This small talk may look insignificant on first sight, but Yacov is a teacher with extremely good reputation and with 25 years of teaching experience. It is hard to believe that such a teacher believes that if a topic is difficult for the students it is because the teacher has not explained it adequately. I suspect that there is some irony here, an irony which is directed to one of the public opinions about mathematics. Namely, mathematics is difficult because it is taught inadequately. Thus, mathematics teachers are held responsible for the present situation in mathematics education. It is something quite unpleasant to face. The teacher was, perhaps, looking for a way to get it out of his system on one hand, and on the other hand, looking for reassurance(from me and from his colleagues) that this is not the case.

During the discussion that took place after that I asked: *Why don't you simply ask the students about their difficulties?* Shoshanah: *Because they will tell you everything is difficult. I have tried it many times. An entire proof is written on the blackboard, filling it completely and I am asking: is anything unclear? And they respond: everything.*

This is also typical to the teacher - student communication in many mathematics classes. There is no meaningful dialogue between the two. As a result of that and the fact that mathematics education theories are not used, teachers are, very often, left with some of the common (and not necessarily helpful) devices as explaining again and again and assigning lot of exercises (see above). Here are some more excerpts from the discussion about the topics.

Carmela: *After many years that I haven't faced any difficulty in teaching trigonometry I had some serious problems with my 11-th graders last year. As a matter of fact - a good class. Probably, the weirdness of a new stuff. This year, it is not a problem for them anymore. I asked them: What happened to you last year that you cried so much ?* I: *What was so difficult for them in your opinion?* Carmela: *I think this is the well known fear of anything new...What did I do? I explained again and again. If in previous years I first explained A, then B and finally C - last year I first explained B, then C and finally A; all the possible permutations. I did a lot of exercises... Why was it difficult? I do not know.* I: *How do you teach it?* Carmela: *I start immediately with the functions, the sine, the cosine and so on. When I realized it was difficult for them I switched to the triangle. Then they told me: Yes, but here it is in a triangle. They had this difficulty; I do not know why.* I: *Hasn't this happened to you also in the past?* Carmela: *No. Not that I remember.* I: *And how do you explain this?* Carmela: *I do not know. I am asking you.*

Carmela is a department head at her school with 28 years of teaching experience, extremely devoted to her students and colleagues. Yet, if you examine carefully her above comments you discover lack of reflection and lack of resources to solve some teaching problems. In order to make her students understand she tries various orders of presenting the materials. It seems that one order is better than the other but she cannot explain why. The fact was that this year she taught trigonometry in the functional approach. Just a simple task analysis, which is so common in mathematics education research, tells you immediately that such an approach would be extremely difficult for the students. Carmela realized it after she tried it and then she regressed to the old approach - the triangle. She has taught trigonometry in the new approach for a couple o years. It is a little bit surprising that she did not notice the students' difficulties with this approach earlier. But it is also quite typical to situations when teachers present to the students something with which they are not familiar. They are so busy with their own difficulties that they cannot notice their students' difficulties.

This part of the discussion reminded some teachers their mathematical experience as high school students.

Gadi: *I remember myself as a high school student, in my first trigonometry lesson I got into a mental block out of which it took me an entire year to get. And I was not a weak mathematics student.* Carmela: *Do you remember why?* Gadi: *They blew us with seven Latin words simultaneously: trig, sine, cosine, tangent, cotangent. secant and cosecant.* Ricki: *This was exactly what they did to us.* Gadi: *My main problem was that I did not understand what was important and what was unimportant. We got an infinite amount of exercises. Infinite amount of trigonometric identities. I proved some identities. So what? I knew more or less what they expected from me. I did more or less what I was expected. I got reasonable marks and I understood nothing. Just nothing.*

I: ... *You said as a high school student you knew how to calculate the sine of any given angle but the concept of sina was not clear to you?*

Gadi: *That's correct.*

Note that in the last part of the discussion teachers talk to each other and not only to me, as happened in previous excerpts. This development I considered as one of the achievements of the workshop. Usually, teachers do not talk to each other on this kind of issues. Also note that these particular remarks make the teachers think about how some of their students feel when learning some mathematical topics. There is an illumination here that, perhaps, I am confusing my students now the same way my mathematics teachers confused me in the past. Isn't this, together with all the above, a solid ground for at least occasional frustration?

2. The humiliation

In a later part of the discussion the issue of in-service training was brought up.

Carmela: *In in-service training I have always had the feeling that the old teachers are treated with disrespect. Look at the kind of mathematics they teach! They want us to emphasize understanding. But the concepts are so difficult. No student can really understand it completely. We were willing to teach in this approach but it simply did not work out. So, please, don't tell us that we are just "old cars" that can run only this way.*

Ricki: *This attitude of disrespect we sense in all the in-service training conferences. The "big bosses" sit there on the stage and broadcast: you really do* *not know mathematics. All you do is to prepare your students for the matriculation exams.*

At this point, I really think, interpretations are superfluous.

The in-service training led us to the university pre-service training. Ayalah was arguing that her university training did not prepare her to cope with the new mathematics curriculum. Here are some more excerpts which do not need interpretations.

I: *I think I have understood you. You are claiming that as far as the new curriculum is concerned you haven't got the rationale, the conception, spirit and goals. Isn't it?* Ayalah: *For instance, I haven't studied physics and therefore I do not know at all what vectors are.* I: *Don't you?* Ayalah: *O.K. I saw it in linear algebra. But there, they never drew any picture on the blackboard. I have never seen the parallelograms with the vectors on them. This is something which is entirely new for me. I feel that it is wrong for me to teach this stuff if my knowledge goes only as far as solving the exercises at the end. I require from myself to learn something which will help me to understand what this vector is. When it is used? What exactly do they do with it? Technically I can be a mathematics teacher without knowing physics, but I feel that I have to study it in the future.* I: *Do you have any complaints against the mathematics department that did not give you the appropriate training? Do you think that certain topics should have been included in your B.Sc. program and have not been included there? Or it didn't occur to you that you can criticize?* Ayalah: *I have no problem criticizing... I came to the university from a school in which they do not teach 5 unit mathematics... first year university - I did not understand what it was all about . A year and a half I was in a complete shock... All the clever guys sat in the first line and had a dialogue with the lecturer. My criticism against the university is that the enrollment requirements for the mathematics department are very low. The university knows that drop out is 50%. But why not collecting a one year tuition from so many students. And what do they care about 200 students sitting in one lecture hall? This fact gave me the chance to run away. It is not one to one situation; a teacher and a student. If I were in a smaller group I would have felt some pressure. The pressure of the lecturer who knows exactly what I know and what I don't, and also my own pressure to understand everything which is require for a particular lesson. If things were like that at the university I would have come out of it with a different type of knowledge.*

This is about content knowledge. The attitude to pedagogical knowledge is expressed in the following excerpt.

I: *What about (preservice) teacher training. It has one course which is called [illegible] didactics course.* Teachers (jiggling): ... I: *Did it have any contribution [illegible] daily practice?* Daphna: *I did not go through the teacher training program [illegible] got a teaching permit. I had to take an examination in order to get it. Didactics [illegible] was included in the exam materials. I got a pile of books to read. I showed them I* *knew how to pass an exam... That's it.* I: *Do you remember any book from that pile?* Daphna: ... I: *Just one.* Daphna: *No. And my marks were high, above 90.* Very often, teachers are criticized by the system (universities, parents, the educational administration). It seems to me that the above excerpts leave no doubt about the address to which this criticism should be forwarded.

3. The hope.

There is no universal recipe for hope. Different people might have totally different views about it. Thus, I do not know whether my view here will be widely acceptable or acceptable at all by other mathematics educators. I believe it was expressed in much better way that I can think of in The Board of Education's Handbook Suggestions for Teachers (1937), an English document quoted in Shuard&Quadling (1980, p. 2): "The only uniformity of practice that the Board of Education desire to see ...is that each teacher shall think for himself and work out for himself such methods of teaching as may use his powers to the best advantage... of his school." Teachers, very often against their own will, have become messengers of systems with which they do not identify. These systems do not give them enough freedom to express themselves as human beings. Teaching is a creative profession. The educational systems should think how to encourage teachers' creativity (without losing their control, I know). It is not a simple task . However, the occasions in which teachers expressed satisfaction and enthusiasm about their work were the occasions where creativity was involved. They were related to mathematical activities, to working sheets or to a new way of presenting a concept, all originated by the teachers. How this can be, at least partly, obtained is a topic for another paper.

References

Bottino, R. & Furingheti, F. (1994). Teaching mathematics and using computers: Links between teachers' beliefs in two different domains. The Proceedings of the PME 18, Vol. II. II, pp. 112 - 119.

Boufi , A(1994). A case study of a teacher's change in teaching mathematics. The Proceedings of the PME 18, Vol. II, pp. 120 - 127.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. Educational Studies in Mathematics, 21, 521 - 544.

Polya, G. (1957). How to solve it. Doubleday Anchor Books.

Schoenfeld, A. (1985). Mathematical Problem Solving. Academic Press, Harcourt Brace Jovanovich.

Shuard, H. & Quadling, D (Ed.) (1980). Teachers of Mathematics-Some Aspects of Professional Life. Harper &Row.

Vinner, S. (1995). Teaching Mathematics as an educational Task. The Proceedings of PME 19, Vol. 3, 328 - 335.

TEACHERS' KNOWLEDGE OF PUPILS' ERRORS IN ALGEBRA

by Ellam K Wanjala, Kenyatta University, Kenya,
and Anthony Orton, University of Leeds, UK.

This paper outlines part of a major study into Kenyan pupils' errors in algebra and their teachers' strategies for dealing with those errors. Here, the nature and selected results of an investigation into teachers' knowledge of pupils' errors are reported. The teachers were required first to place similar algebra tasks into expected order of difficulty, then to predict likely pupil errors, and finally to suggest strategies for helping the pupils. The results indicate that many teachers are aware of likely errors, but that many other teachers are lacking in essential knowledge.

Introduction

When pupils are presented with an algebra question such as:

Task 1 *Simplify where possible*
(i) $3x + 8y + x$
(ii) $(a - b) + b$
(iii) $3a - (b + a)$
(iv) $2a + 5b$

do their teachers know what are the relative difficulties of the four subtasks? Indeed, do the teachers know what the likely errors are? Can teachers accurately predict the errors which pupils are likely to make on subtasks like "Simplify 3a - (b + a)" and do they have strategies for dealing with the errors?

This paper is based on some of the findings of a major study carried out in Kenya, concerning both pupils' errors in what might most simply be described as traditional school algebra, and also with teacher strategies in identifying and counteracting the errors. A major part of the study consisted of a detailed analysis of the range and types of errors committed by 900 pupils from seven different secondary schools of three different types and across three forms (i.e. 'year' groups or standards). This kind of study has already been carried out in many countries around the world, but it was considered necessary to collect data concerning errors committed by Kenyan children and not to base a study of teacher awareness on data from other countries. The teachers themselves completed a short questionnaire in three parts. In the first part (**Comparison**), they were asked to place subtasks, such as those in Task 1, in order of difficulty, in the second part (**Prediction**) they were asked to predict likely pupil errors in a task such as (iii) above, and in the final part (**Analysis**) they were expected to identify errors and suggest remedial teaching strategies, having first been provided with examples of pupils' erroneous solutions. It is the analysis of the outcomes from the teacher questionnaire which provides the data for this paper.

Errors and misunderstandings generated by school pupils have been researched and reported widely. Examples of earlier studies include those by Collis (1975), Küchemann (1981), and Booth (1984). Driscoll (1982) wrote a useful account of how research findings in the field of learning algebra might inform teachers and lead to better understanding in pupils. Herscovics (1989) detailed what he described as cognitive obstacles which pupils encounter in learning algebra. These are but a few examples of the many studies which enabled Warren (1992) to suggest five major areas of difficulty in traditional school algebra, namely the variable concept, the visual syntax of algebra, the concatenation of algebraic expressions, the changing nature of the equal sign and the manipulation of symbols. The knowledge which is now available to the community of mathematics educators will not benefit the pupils, however, unless secondary school mathematics teachers generally are aware of these likely errors and alternative conceptions. Most teachers do not appear to read even what for them are the most relevant research papers, or even professional journals which might contain research articles. The most likely first source of knowledge about pupils' learning difficulties, for most teachers, is therefore likely to be the behaviour they observe in their pupils.

Comparison

The basis for the consideration of teacher responses in this section was the facility levels which resulted from the pupil test data. These facility levels were determined both by using mean score on the item, and response level, where the response level (R) is defined as $R = (P_H + P_L)/2$ (Youngman, 1979), and where P_H is the proportion of pupils in the high-scoring group responding correctly to the task, and P_L is the corresponding proportion of pupils in the low-scoring group. The high- and low-scoring groups consisted of pupils whose total scores on the written test fell within the top and bottom twenty-five per cent, respectively. The decision to use two methods was mainly because that provided a means of counter-checking the order of difficulty. It was acknowledged that the two methods could lead either to concurrent or contradictory results, but that either outcome would inform the study. In fact, the results were never contradictory. A further statistic was computed, namely the discrimination (D), defined as $D = P_H - P_L$ (Youngman, 1979). The purpose of the discrimination was to determine the suitability of each item as a test item. The higher the discrimination, the better suited the item for testing.

It was realized that arranging subtasks in order of increasing difficulty would lead to many possible responses from teachers. Hypothetically, there would be 24 possible arrangements of the subtasks of Task 1. Subsequent questions, with five and six parts, respectively, would lead to 120 and 720 possible arrangements! It was therefore clearly necessary to categorize responses in some way, and this categorization was based on the degree of empathy with the actual responses of the pupils. The category described as **Strong Empathy** (SE) consisted of those responses that were considered to match closely the actual order of difficulty revealed by pupil responses, and was generally based on the fact that the easiest were put first and the hardest were put last. The **Moderate Empathy** (ME) category included responses in which the easiest were placed before the hardest, but which otherwise did not meet the stricter requirements of Strong Empathy. The **Weak Empathy** (WE) category then included all those responses in which the hardest were put before the easiest. Results from two of the tasks are reported here.

The facility levels and discriminations for Task 1 (quoted at the beginning of this paper) are shown in Table 1a, and the frequencies of responses in the categories of empathy are shown in Table 1b. The discrimination values were all within the desirable limits. There were only two types of response which were considered to be good enough to be classified as SE, namely iv-i-ii-iii and i-iv-ii-iii.

Table 1a : Facility levels for Task 1

Item	iv	i	ii	iii
Response level	0.74	0.66	0.45	0.39
Mean score	0.81	0.75	0.45	0.36
Discrimination	0.35	0.20	0.87	0.72

Table 1b : Frequencies of categories for Task 1

Degree of empathy	SE	ME	WE	NR*	TOTAL
Number of teachers	47	7	12	1	67
Percentage of teachers	70.1	10.4	17.9	1.5	100**

* NR denotes no response ** denotes rounding

It is clear from these data that a large proportion of the teachers seemed to acknowledge the fact that the introduction of brackets in algebraic expressions increases the level of difficulty for the pupils. Indeed, their placing of ii before iii indicated that these teachers perceived that the introduction of a negative sign before a bracket compounded the difficulty that pupils experience with brackets. It is also clear that as many as 17.9 per cent of the teachers were unaware of the difficulties created by the introduction of brackets, let alone the negative sign before a bracket. It would appear that this category of teachers would not have a basis for an appropriate sequencing of teaching materials and would be limited in the degree of help they could provide to pupils.

Task 2 *Simplify where you can:*

(i) $\frac{ax}{bx}$ *(ii)* $\frac{a^2}{a}$ *(iii)* $\frac{a}{2(a+b)} + \frac{b}{2(a+b)}$

(iv) $\frac{a}{a^2}$ *(v)* $\frac{a+x}{b+x}$ *(vi)* $\frac{1}{3x} + \frac{2}{x}$

The facility levels and discriminations for Task 2 are shown in Table 2a, and the frequencies of responses in the categories of empathy are shown in Table 2b. The discrimination values were again all within the required limits. The SE category consisted of responses in which the first three places were filled with items from the easiest band (i, ii and iv), in any order, and the hardest (iii) was placed last. The WE category consisted of all the responses in which the hardest was placed before any of the items from the easiest band. The ME category then consisted of all the other responses.

Table 2a : Facility levels for Task 2

Item	i	ii	iv	v	vi	iii
Response level	0.81	0.70	0.52	0.49	0.38	0.29
Mean score	0.82	0.80	0.67	0.49	0.45	0.27
Discrimination	0.25	0.45	0.80	0.80	0.72	0.58

Table 2b : Frequencies of categories for Task 2

Degree of empathy	SE	ME	WE	NR	TOTAL
Number of teachers	25	31	5	6	67
Percentage of teachers	37.3	46.3	7.5	9.0	100**

One must acknowledge first that it seems natural that it would be more difficult to arrange six subtasks in order of difficulty than four. Also, any comparison of results between Tasks 1 and 2 must take into account the classifications used to define the levels of empathy. Nevertheless, it seems that only 37.3 per cent of the teachers were able to sequence the subtasks of Task 2 in a near enough order to reflect the learning difficulties experienced by pupils. The teachers in this category seemed to be aware that algebraic fractions involving expressions that contain more than one term present more difficulties than single term fractions. They were certainly also able to perceive the complexity of subtask iii. The largest group of teachers was the ME category. In fact, 29 of the 31 teachers in this category placed the three easiest items in the first three places, but placed the hardest (iii) before either or both of the middle band subtasks (v and vi). All of the teachers in the WE category placed item iii before any of the three easiest subtasks, indicating a poor understanding of the comparative difficulties. Although there was fortunately only a small proportion of the teachers in the WE category, when combined with the NR group they form a similar-sized group to those who were weak in assessing difficulty levels in Task 1. Conclusions from the Comparison tasks suggest that a considerable number of teachers had some appreciation of difficulty levels, in some cases a good appreciation, but that a worrying subgroup of nearly one fifth of the teachers had little idea.

Prediction

The analysis here was based on first determining the frequencies of the errors which teachers had predicted, and then categorizing these responses according to their status in terms of what the pupils did. In order to reduce the number of errors predicted by teachers, only those with a frequency of at least 3 per cent were retained in the analysis. Three main categories were used for the responses, referred to here for convenience as **Coincident**, **Unnoticed** and **Hypothetical**. Coincident errors were those which teachers predicted which were indeed committed by the pupils, unnoticed errors were those which were committed by pupils but which did not feature in the predictions of the teachers, and hypothetical errors were those predicted by the teachers which no pupils committed. Other categories which were used to complete the classification were **Vague responses** and **Nil responses**. A selection of the tasks used together with the corresponding results are now described.

Table 3 : Frequencies of predicted errors in Task 1 (iii)

Error	Number of teachers	Percentage of teachers
4a - b	54	80.6
2a + b	3	4.5
4ab	3	4.5
Vague	7	10.4
Nil	1	1.5

Two errors were coincident in Task 1 (iii), namely 4a - b and 2a + b. These two errors reflect the difficulty pupils have in relating the negative sign to the bracket, and it is clear many teachers are aware of the problem. There were, however, also five unnoticed errors, namely $3ab - 3a^2$, $-3ab - 3a^2$, 3ab - ba and 3 - b. The first three of these were seemingly due to the association of removal of a bracket with multiplication, while the last two involved lack of basic knowledge of algebraic notation and convention. In fact, these last two were committed by 9 per cent and 5 per cent of the pupils, respectively, suggesting they are common enough errors for teachers to need to be aware of them. Only one hypothetical error was recorded, namely 4ab.

Table 4 : Frequencies of predicted errors in Task 1 (iv)

Error	Number of teachers	Percentage of teachers
7ab	47	70.1
7a	8	11.9
7b	7	10.4
7(a + b)	5	7.5
10ab	4	6.0
7a + b	3	4.5
2 + 5	3	4.5
Vague	3	4.5
Nil	6	9.0

The coincident errors in Task 1 (iv) were 7ab and 10ab. Indeed, earlier research has indicated that the concatenation error 7ab is very common (see Booth, 1984). There was only one unnoticed error, 7 + ab, which is again associated with poor knowledge of algebraic notation and convention. Five hypothetical errors were predicted by the teachers, namely 7a, 7b, 7(a + b), 7a + b and 2 + 5. Some of these were predicted by quite large proportions of the teachers, and altogether the five responses indicate some lack of awareness of likely errors on the part of the teachers.

Table 5 : Frequencies of predicted errors in Task 2 (iv)

Error	Number of teachers	Percentage of teachers
1/2	26	38.8
a	10	14.9
1/a	3	4.5
a/a^2	2	3.0
1	2	3.0
Vague	6	9.0
Nil	10	14.9

Coincident errors on Task 2 (iv) were ½, a and 1. However, although the error ½ was predicted more frequently than any other, it was only committed by 3 per cent of the pupils. The most common pupil error was 'a', which was predicted by only 14.9 per cent of the teachers. There is thus the indication that many teachers are well aware of some of the likely errors, but are not always aware of their relative frequency. The suggestion that teachers are only aware of some of the errors is supported by the existence of five unnoticed errors, namely -a, a/2a, 1/-a, $1/1^2$ and 2. Two hypothetical errors were suggested by the teachers, and these were 1/a and a/a^2. Given that 1/a is the correct answer, and a/a^2 is the original question, these must both be considered strange responses which perhaps suggest either lack of understanding of the question, or even lack of understanding of the mathematics.

Task 3 *Write $(3x + 2)^2$ without brackets.*

Table 6 : Frequencies of predicted errors in Task 3

Error	Number of teachers	Percentage of teachers
$9x^2 + 4$	27	40.3
$6x + 4$	20	29.9
$3x^2 + 4$	10	14.9
$9x + 4$	6	9.0
$3x + 4$	5	7.5
$9x^2 + 12x + 4$	3	4.5
10x	2	3.0
Vague	3	4.5
Nil	1	1.5

The coincident errors on Task 3 were $9x^2 + 4$, $6x + 4$, $3x^2 + 4$ and $9x + 4$. The first of these was both the most commonly predicted and the most frequently committed by the pupils. In fact, it is such a well known error that it is surprising that more teachers did not predict it. The error $6x + 4$ was also very common. Hypothetical errors were $9x^2 + 12x + 4$, $3x + 4$ and $10x$. Here, we find the correct response again being included as a predicted error, which is difficult to understand. Presumably, $3x + 4$ was suggested on the basis of squaring only the 2, that is, ignoring the brackets. There were no unnoticed errors on this task.

Analysis

In this section, teachers were provided with typical pupil questions together with a corresponding incorrect solution. They were asked to (a) identify what the pupils had done wrong, and (b) suggest ways of helping. An example of a question and its corresponding incorrect solution is given as Task 4.

Task 4

If $n - 246 = 762$ *then* $n - 247 =$

Pupil solution: 763

This section of the teacher questionnaire was the most complex to analyze, because of the relatively open nature of the tasks and the corresponding wide variety of responses obtained. Using procedures described by Bliss et al. (1983), four categories of response were defined, namely 'error identified', 'answer given', 'irrelevant response' and 'no response'. The 'error identified' category represented cases in which the teachers were conceptually inclined, that is they were diagnostic in their approach and were interested in the nature of the misconceptions which led to the response. The 'answer given' category, on the other hand, represented cases where the teachers only seemed to be interested in the correct solutions. As regards suggestions, again four categories were defined, namely 'pupil based', 'subject based', 'irrelevant suggestion' and 'no suggestion'. The 'pupil based' suggestions represented cases in which teacher strategies were sensitive to the pupils' difficulties. They often made use of relevant prior and perhaps more elementary but relevant knowledge of the pupils. On the other hand, the 'subject based' suggestions focused on the accuracy of the response but were insensitive to the pupils' difficulties. At this stage it is possible to report only on the frequencies of the various categories. Table 7 shows the relative frequencies of the categories on error identification, and Table 8 shows the relative frequencies of types of suggestions.

Table 7 : Relative frequencies of categories on error identification

Classification	Number of cases	Relative frequency
Error identified	111	0.33
Answer given	78	0.23
Irrelevant response	89	0.27
No response	57	0.17
Total	335	1.00

Table 8 : Relative frequencies of the types of suggestions

Kind of suggestion	Number of cases	Relative frequency
Pupil based	24	0.07
Subject based	87	0.26
Irrelevant suggestion	104	0.31
No suggestion	120	0.36
Total	335	1.00

The indications are that in about one third of the cases the teachers were able to identify an error, in about one quarter of the cases the teachers just gave a correct answer, in over one quarter of the cases the teachers made irrelevant suggestions, and in nearly one fifth of the cases the teachers were unable to give any response. Initial consideration of the individual questions reveals that the teachers could identify errors more easily in some situations than in others. In Task 4, errors were identified with a relative frequency of 0.48, whereas in two other questions figures of only 0.15 and 0.19 were obtained. The suggestion here is that many teachers do not see such questions, and indeed possibly algebra as a whole, as having to convey any meaning to pupils. Rather, the emphasis seems to be on symbol manipulation. It seems that many teachers need to be persuaded that the teaching emphasis should be on making algebra as meaningful as possible to pupils.

References

Bliss, J., Monk, M. and Osborn, J. (1983) *Qualitative Data Analysis for Educational Research*. London: Croom Helm.

Booth, L. R. (1984) *Algebra: Children's Strategies and Errors*. Windsor: NFER-Nelson.

Collis, K. F. (1975) *A Study of Concrete and Formal Operations in School Mathematics*. Hawthorn, Victoria: Australian Council for Educational Research.

Driscoll, M. (1982) *Research Within Reach: Secondary School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Herscovics, N. (1989) 'Cognitive obstacles encountered in the learning of algebra'. In S. Wagner and C. Kieran (eds) *Research Issues in the Learning and Teaching of Algebra*. Reston, VA: National Council of Teachers of Mathematics / Erlbaum.

Warren, E. (1992) 'Algebra: Beyond manipulating symbols'. In A. Baturo and T. Cooper (eds) *New Directions in Algebra Education*. Red Hill, Australia: Centre for Mathematics and Science Education, Queensland University of Technology.

Youngman, M. B. (1979) *Analysing Social and Educational Research Data*. London: McGraw-Hill.

CHILDREN'S STRATEGIES FOR COMPARING TWO TYPES OF RANDOM GENERATORS

Jenni Way

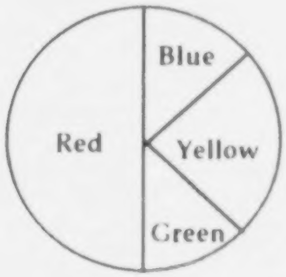
University of Western Sydney, Nepean.

This paper reports on one particular interview task that required children (aged 5 to 12 years) to compare the structure of one form of random generator to another. Their strategies can be classified into four distinct categories which are age related. The most effective strategy, in terms of correct solutions, involved fractional thinking. The strategies are discussed in terms of some recent research on fractional and proportional thinking.

Introduction

The task-based interviews reported in this paper refer to a particular task from one component of an ongoing exploratory study of children's understanding of probability. This 'Transfer task' was designed to see if the children were able to relate the structure of one type of random generator to another. Although spinners and coloured objects in containers are two random generators commonly used in probability studies, little work has been reported on children's thoughts about the relationship between the two. One study in progress (Truran, 1994) suggests that children perceive the behaviour of various random generators to be quite different even though they are mathematically the same. The 'Transfer Task' provided a small opportunity to explore this aspect of probabilistic thinking.

The children first played a game involving the drawing of small coloured bears from a box, which gave them experience with a numerical model random generator. Then they played a 'Racing Car' game involving the use of four different spinners, which gave the children experience with an area model random generator. The 'Transfer task' required the children to mix the two games by placing coloured bears into the box, ready to play the 'Racing Car' game. They were asked to match the structure of three types of spinners, but only the biased spinner with the proportions 4:2:1:1 is reported here (see diagram below). Thus, the children were required to consider the relative proportions of the spinner's sections and decide on numerical values for its various sized sections.



Red: Yellow: Blue: Green = 4:2:1:1

Results

So far the data for 32 children has been analysed. Table 1 shows the number of children of each age and the number of children that achieved correct solutions, with the percentage of that age group in parentheses. While many of the children came very close to a correct solution, only 18.75% of the 32 children were actually correct. No children under the age of 9 years were correct.

Table 1: Age Group/ Number Correct

Age (years)	Number of children	Number with a correct solution
5	3	0
6	4	0
7	3	0
8	5	0
9	5	1 (20%)
10	4	1 (25%)
11	5	2 (40%)
12	3	2 (66%)

The children's solutions and strategies were examined for similarities and differences, then grouped together accordingly. Four distinct categories of strategies became apparent; Non-comparison, Measurement, Ordering and Fractional. Descriptions of the four strategies follow. In the examples given, C = Child, I = Interviewer and the four colours are represented by their first letter. Anything appearing in parentheses has been added by the interviewer to assist interpretation.

Strategy 1: Non-comparison

This strategy, generally restricted to the youngest interviewees, lacks comparison of the sizes of the four colour regions. All of the responses in this category included all four colours, usually in equal amounts (typically two or three bears), but sometimes in uncounted handfuls. Some children placed a larger number of Red bears in the box, acknowledging the dominance of the Red portion, but not indicating the ability to make more specific relative comparisons. Most of these children were unable to explain their strategy in a logical way. A few children, who at first seemed to be applying this strategy, revealed through their explanation that they had assumed they were supposed to construct a 'fair' sample space, and after further explanation of the task, changed their solution.

Example 1 - Angela (5:6 yrs)
Solution: A handful of each colour, which was about 4 or 5 bears.

C: Handful of each colour - 'cause I looked at all the colours.
I: Did looking tell you how many bears to put in?
C: 3
I: Why three?
C: There's 3 corners (on each sector).

Example 2 - April (6:11 yrs)
Solution: All R 2Y 2B 2G
(At first had 2 of each colour)
I: Which car would probably win using the spinner?
C: The red one.
I: Would the red car win using your box?
C: Mm....well. (Tipped in all the Red bears.)

Strategy 2: Measurement

These responses involved using the bears as a unit of measurement to estimate the 'size' of each coloured portion on the spinner. This was typically done visually at first, and then in a concrete demonstration when an explanation was sought by the interviewer. The younger children tended to apply linear measurement, such as lining up the bears around the outer curved edge of the spinner sectors, or across the width of each sector. This resulted in numerical values for Red ranging from 3 to 10. The older children tended to think in terms of area and some seemed to be also using some informal proportional thinking to 'scale down' the number of bears they had to place into the box (see Example 3 below). The oldest child (10:4 yrs) achieved a correct solution and several others were very close. Several responses were allocated this category because on the surface they appear to fit, but they contained hints of fractional thinking. For example, Alyssa (6:11 yrs) kept mentioning "slices in a pie" when trying to explain her choice of numbers and although she was referring to counting, may have been trying to describe the eighths that were the basis of the spinner's construction. Unfortunately, these children were unable to provide further explanation of their thinking and did not choose to demonstrate physically. It is possible that they had some intuitive ideas that they could not put into words, or resorted to simple measurement words, such as 'fit', to approximate their meaning.

Example 1 - Jill (8:9 yrs)
Solution: 5R 3Y 2B 1G
C: 'Cause the Green's got the less area than the Blue has, and then the Yellow's got the less area than the Red has, it's got the most area.
I: Why 3 Yellow?
C: It sort of seems that would fit there (showed bears laying on their backs on the spinner).

Example 2 - Allan (7:6 yrs)
Solution: 6R 2B 3Y 2G
C: I'm measuring the colours on the spinner. (Laid bears on backs shoulder to shoulder in a straight line across each sector).

Example 3 - Anne (8:7 yrs)
Solution: 5R 3Y 2B 2G
C: Because 5 bears would be more....it would be about 10 bears on this (R sector of spinner) but I didn't put 10 because there wouldn't be any room for the others (in the box). 2G would go there (on spinner) and 3 of Y fits there - I sought of guessed how many.

Strategy 3: Ordering

In this strategy the children typically assigned a number to the Red portion, then assigned the next biggest portion (Yellow) one less bear, then assigned either the Blue or Green the next two numbers in the descending sequence (eg. 4, 3, 2, 1). This was despite the fact that the Blue and Green sectors were the same size. Some children acknowledged this by assigning the same number of bears to both, while others showed indecision about this equality. The desire to follow the descending pattern seemed to override the visual message of equal area. Some children began by assigning one bear to the smallest sector, then used an ascending-by-ones sequence. One child assigned the largest sector (Red) 1, then ranked the other sectors 2, 3 and 4 to indicate their decreasing size. While none of these responses produced correct proportions, some came very close, for example: 3R 2Y 1B 1G or 4R 3Y 2B 2G.

Example 1 - Aaron (11 yrs)
Solution: All R 4Y 3B 2G
C: Because that (Red) was half, that has the biggest chance so I put them all in, Yellow is the next biggest so I put 4 in to make it have another chance, I put 3 or 2 of the Blue and Green 'cause they're about the same and it gives them less of a chance than the R and the Y.

Example 2 - Arthur (10 yrs)
Solution: 6R 5Y 3B 4G
C: I think Blue and Green are a bit the same - or Green's a little bit bigger (took out a Blue). The Red was the biggest, the Yellow the second biggest, the Green a little bit fatter than Blue.
I: Why 6 Red?
C: Don't know.

Example 3 - Alison (10:6 yrs)
Solution: 3R 2Y 1B 1G .
C: Red is the most, them two are the same and Yellow is the second most, so I just gave 3 to the highest and 2 to that one (Y), and 1 to those two because they're the same.

Strategy 4: Fractional

The children applying this strategy used fractional concepts in some way to determine the number of bears to put in the box. They clearly considered the relative sizes of the sectors in a numerical way and were concerned with finding precise solutions rather than estimations. Most responses of this type were numerically correct. Within this category a number of approaches were evident, depending on what base unit was selected for halving or doubling, thus beginning with the largest sector (Red), the middle-sized sector (Yellow) or one of the smallest (Blue or Green). Four children based their reasoning on a value for the whole circle and two of these worked with equivalent fractions.

Example 1 - Amie (9:7 yrs)
Solution: 4R 2Y 1B 1G
C: Well I worked out that Red is half the circle and I worked out that Yellow equals 2 and Red must equal to 4 'cause it's twice the size of this, and I knew that Green and Blue were 1 because it's half the size of this (Yellow) which is 2 bears.

Example 2 - James (12:7 yrs)
Solution: 4R 2Y 1B 1G
C: Made a number for the spinner and halved it for Red, made that (Y) quarter so it's 2.
I: What number did you make the spinner?
C: 8
I: Why?
C: It's even.

Example 3 - Jack (11:8 yrs)
Solution: 5R $2^{1}{}_{2}$Y $1^{1}{}_{4}$B $1^{1}{}_{4}$G
C: Five Red...(Long thinking time)... I'm going to have to chop some in half (meaning the bears). 5 Red make up a half, $2^{1}{}_{2}$ for Yellow, $1^{1}{}_{4}$ for Blue and Green.
I: If you changed the number of Red could you work it out so you could actually put the bears in the box?
C: (Pause) Probably could. (Obviously reluctant)
I: Why 5 Red then?
C: Scaling down from 100%, make it 10 bears to go in (indicated total circle), so half is 5.

Example 4 - Justin (9:6 yrs)
Solution: 4R 2Y 1B 1G
C: I've divided into eighths. 1 bear demonstrates 1/8, Y is 2 because it's 1/4, and 2/8 makes 1/4. Half is the same 1/2 of 8/8 which would be 4, so I've put in 4 (R).

Table 2 gives the number of children in each age group who used each type of strategy and shows the number who achieved a correct solution. The eleven-year-old in the non-comparison column had difficulty in understanding the task and changed his answer a few times. Although there is substantial overlap between age groups, it can be seen that the type of strategy used is related to the age of the child. This suggests a developmental sequence of understanding.

Table 2: Age Group/ Strategy / Correct Solution

Age	**Non-comparison Strategy**		**Measurement Strategy**		**Ordering Strategy**		**Fractional Strategy**	
	Correct	Incorrect	Correct	Incorrect	Correct	Incorrect	Correct	Incorrect
5	0	3						
6	0	2	0	2				
7			0	3				
8			0	4	0	1		
9			0	1	0	2	2	0
10			1	0	0	2	0	1
11	0	1			0	2	2	0
12							2	1
Total	6		11		7		8	

Discussion

One aspect of mathematical thinking that is crucial to development of a mature understanding of basic probability is that of fraction, ratio and proportion. There has been some debate over what age this type of thinking becomes functional and also over what types of research tasks best access this thinking (eg. Davies, 1965; Chapman, 1975; Hoemann and Ross, 1971; Yost, Seigel & Andrews, 1962). Piaget and Inhelder (1975) considered the concept of proportion to be a feature of Formal Operational thought, achieved at a time when children or adolescents are able to deal logically with abstract relations. The results from the 'Transfer Task' support the developmental idea, with only the older children applying fractional thinking and achieving precise solutions. However, the nine and ten years olds would not usually be classified as formal operators.

The fact that the structure of the spinner used in the 'Transfer task' can be easily described in terms of halves must have influenced the thinking of the children employing the Fractional strategies. Some of the children, particularly the younger ones, would probably have found a more complex spinner structure much harder to work with. The significance of the perception of 'half' for children making proportional judgements has been highlighted by some recent studies, such as Spinillo (1995) and Wantanabe (1995).

The recent work of two researchers provides two interesting viewpoints from which to examine the 'Transfer task' responses. Firstly, the work of Confrey (1995) focuses on 'splitting' as basis for developing number operations in contrast to 'counting' as a basis for number operations. Splitting structures, such as halving and doubling, involve one as the origin, lead to multiplication and division as the key operations and use ratio as the unit. Counting structures have zero as the origin, addition and subtraction as the key operations (subsequently multiplication as repeated addition and division as repeated subtraction) and the basic unit is one (Confrey, 1995. p.7). The Measurement Strategy responses are clearly based on counting, using one (bear) as the unit, with no involvement of fraction or ratio. The Fractional Strategy responses clearly illustrate halving and doubling structures as a basis for reasoning and some explicitly use a fraction as a unit. The Ordering Strategy does not really fit into either approach, because although a counting sequence is a feature of the children's thinking, the numbers are used as a ranking system rather than applying one as a unit.

Wantanabe's (1995a & 1995b) work has been primarily concerned with the relationship between a child's coordination of units and the understanding of simple fractions. While the coordination of units has typically been examined in terms of whole numbers, the understanding of fractions can also involve the coordination of two different units. "For example, 3/4 is a collection of three units of 1/4 of one unit" (Wantanabe, 1995b, p161). A liberal interpretation of the idea of the coordination of units allows one to view the 'Transfer Task' in a new light. The task essentially requires the children to translate the non-numerical (spatial) units of the spinner into the unit-of-one (bear) model. Using this loose definition, each of the four strategy categories portrays a different method of coordinating these two sets of units. The 'Fractional Strategy' provides the most interesting (and perhaps most valid) application of the coordination of units idea. The children's responses can be sorted into sub-categories according to which sector of the spinner they chose as the spatial unit. Most of the children seemed to be able to coordinate the two set of units well and so reached a correct solution. For example, two children chose to make the smallest sector (B and G) their unit and allocated it one bear. Using visual comparison of the other sectors to the unit sector they determined the operations to be used on the bear unit to find numerical values for these sectors. One child used doubling and the other used multiplication.

One of four children who saw the spinner as a model for one whole allocated eight bears to the whole then used eighths to determine the numerical values for the sectors (See Fractional Example 4 above). This was perhaps the only child to fully coordinate the two units (of 1bear and 1/8 of one circle).

Conclusion

In the 'Transfer Task' most children were able to make sensible estimations which may allow them to make appropriate probability judgements when using the items as random generators in situations where the ratios were not very close. In other words, their estimations would probably not be accurate enough to distinguish between ratios that were very similar, such as only a one eighth difference between spinners or a one bear difference between boxes. The mathematical reasoning applied by the children using a Fractional Strategy suggests that these children might be able to make fairly accurate probability judgements through comparing proportions, though some might have difficulty working with ratios other than halves and quarters. The information needed to test these conjectures can be obtained from future analysis of the other tasks used in the broader study. To obtain a numerically correct solution to the task the children had to construct a fairly sophisticated system of coordinating the systems of units presented by the two random generators.

Although the 'Transfer Task' was a minor component of an exploratory study it exposes the potential of such tasks for revealing the mathematical thinking that children need to be developed if they are to move beyond intuitive understanding and inaccurate estimations when working with the fractions, ratios and proportions required for the numerical level of understanding of probability.

The findings raise a number of questions, the answers to which could be useful for informing teaching practice. If children as young as 6 years are using a Measurement Strategy and children as old as 10 years are still using a Measurement strategy, what teaching/learning could take place to move the children on to a more accurate strategy? Can the Measurement Strategy be used as a basis for developing proportional thinking? If the strategies are age related and hence developmental in nature, why does the Ordering Strategy show no relationship to either the

Measurement or Fractional Strategies? If the Measurement strategy is based on counting structures and the Fractional strategy is based on 'splitting' structures, how is the shift made from one to the other? Are these strategies peculiar to this particular task or would they be induced by other similar tasks?

References

Chapman, R. (1975). The development of children's understanding of proportions. *Child Development, 46,* 141-148.

Confrey, J. (1995). Student voice in examining "splitting" as an approach to ratio, proportion and fractions. In In L Meira & D. Carraher (Eds). *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education.* Vol.1 3-29. Recife, Brazil.

Davies, C. (1965). Development of probability concepts in children. *Child Development, 36,* 779-788.

Hoemann, H., & Ross, M. (1982). Childrens concepts of chance and probability. in C. Bruind (Ed.). *Children's Logical and Mathematical Cognition: Progress in Cognitive Development Research.* New York: Springer-Verlag. 95-121.

Piaget, J., & Inhelder, B. (1975) *The origin of the idea of chance in children.* (Original work 1951). London: Routledge & Kegan Paul.

Spinillo, A. (1995). Can young children learn how to reason proportionally? An intervention study. In In L Meira & D. Carraher (Eds). *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education.* Vol.3 192-199. Recife, Brazil.

Wantanabe, T., Reynolds, A., & Lo, J. (1995a). A fifth graders understanding of fractions and proportions. In In L Meira & D. Carraher (Eds). *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education.* Vol.3 200-207. Recife, Brazil.

Wantanabe, T. (1995b). The coordination of units and understanding of simple fractions: Case studies. *Mathematics Education Research Journal. 7:2.* 160-175.

Truran, K. (1994). Children's understanding of random generators. In J. da Ponte & J. Matos (Eds). *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education.* (p.77) Lisbon, Portugal.

Yost, P., & Seigel, A. (1962). Nonverbal probability judgements by young children. *Child Development, 32,* 769-780.

TEACHING TO CREATE DISCUSSION AS MATHEMATICAL ARGUMENTATION[1]

Terry Wood

Purdue University

Considerable evidence exists as to the pivotal role of discussion in students' development of mathematical understanding and teachers' ways of creating forms of interaction that allow for individual children's reasoning. This paper draws on classroom research to address a more crucial aspect of teaching; the enablement of situations disagreement or confusion among students. The resolution of this not only provokes reflective thinking but resembles processes of mathematical argumentation. Little is known about these interactions, and yet they are central to teaching from a Piagetian constructivist framework. Drawing on analysis of 50 lessons, such discussions are examined. The results reveal the intricate ways the teacher sustains the interaction to allow children's reasoning to prevail, while restricting her own instructive contributions, which enables children to progress themselves in their mathematical thinking.

How pupils come to learn mathematics in school classrooms is a serious question and one which is being attended to by educators and researchers alike. Several, such as Confrey (1994) have examined constructivist theories to offer insights for learning mathematics. Still, even though widespread interest exists in constructivist perspectives, there is not clear agreement as to the meaning of 'constructivism'. This situation is further complicated when attempts are made to bring theory to the practice of schooling. In fact, as Newman, Griffin, and Cole (1989) found, theoretical principles previously successful in examining learning in experimental settings do not necessarily transfer easily to school situations. These difficulties are seen to be principally due to a lack of a clear connection between theoretical constructs developed in experimental settings and their application in the more complex educational settings which have somewhat competing goals.

Nonetheless, a Piagetian constructivist theory continues to attract many teachers who hold an interest in child-centered approaches to learning. Yet, a child-oriented philosophy is not enough, and teachers need to develop an understanding of, and to some extent accept, those constructivist tenets that distinguish Piaget's theory from others. Once understood, it is then possible for teachers to transform these tenets and to generate a distinct form of practice which extends beyond a superficial form of implementation.

This process involves a serious shift in the roles for the teacher and students which includes a renegotiation of the rights and responsibilities which underlie the nature of the interaction that evolves. This change is most clearly seen in situations involving class discussion which center on children offering their thinking and reasoning about mathematical problems. At the most simplistic level, these are settings in which the teacher's role is one of allowing individual children's mathematical thinking and reasoning to come to the fore, while limiting their own instructive contributions.

---

[1] The research reported in this paper was supported by the National Science Foundation under grant RED 9254939. All opinions are those of the author.

Yet, a more complex aspect of these discussions are those instances of disagreement or confusion which arise from the diversity in ideas generated by pupils' thinking. Ideally, these are situations in which children, as a group, are provided the opportunity to resolve their differences. Moreover, to do so, children are seen to engage in a process of resolution which resembles mathematical argumentation. This consists of interactions between the child, as the explainer, and the others as listeners, and involves processes of explaining, challenging, justifying; continuing until eventually resolution occurs. This process is regarded as a fundamental activity of and central to the discipline of mathematics (Devlin, 1994).

However, in reality discussion involving young children in productive argumentation is one of the most challenging and complex for teachers to develop. This is largely due to the fact that success of such interaction depends on teachers developing three essential aspects; 1) children's personal comfort in engaging in public discourse in which disagreements arise, 2) creating an active participatory role for children as listeners, 3) limiting their own contributions while enabling children progress in their thinking. Therefore, it is not surprising to find that most discussions in which mathematical argumentation occurs simply consists of an exchange between the student and teacher, while the remaining children assume the traditional role of passive listeners.

The purpose of this paper, therefore, is to examine the processes of teaching which underlie those discussions in which students engage in the resolution of disagreement or confusion in their mathematical thinking. The paper is directed at two ends; to describe what it is a teacher does during these events to sustain students in the processes of mathematical argumentation, and, to lay bare the manner in which the social processes that underlie this form of interaction are constituted. In order to exemplify these processes, illustrations are drawn from the classroom lessons of one teacher.

First, classroom episodes are presented to illustrate the teacher's role. The particular episodes characterize ways in which the teacher makes use of situations in which disagreements arise and/or students experience confusion in their thinking. The examples reveal the intricate process by which the teacher maintains the interactive framework which sustains children in their respective roles as explainer and 'active listener'. It is in these situations, which children bring their reasoning to the forefront and contribute in the process of mathematical argumentation that allows them to progress themselves in their thinking.

Next, the process is traced by which the teacher establishes the norms that underlie the interaction. This process occurs during the beginning weeks of the school year and illustrates the sensitive manner in which the teacher establishes with the children the distinctive norms which clarify their roles during class discussion. Concurrent with these analyses, insights into the teacher's perceived roles for her students and herself are revealed through comments made during a series of interviews. The end result is to provide insight into the ways in which the teacher has created a social context which is essential to support her recasting of constructivist theory into practice.

THEORETICAL ORIENTATION AND BRIEF BACKGROUND

The class presented is part of a research and development project which began with a constructivist approach to learning influenced by the theories of Piaget and Radical Constructivism as coined by von Glasersfeld (Wood, Cobb, Yackel & Dillon, 1993). The aspect of these theories of importance is not universal stages of development, but rather the fundamental contention that children make *interpretative* constructions and reconstructions in their thinking as they reflect on their activity and that of others. Further, that teaching consists of enabling students to construct personal meaning for mathematics, rather than conveying 'ready made knowledge'.

This theoretical orientation formed the basis for the development of the instructional activities and the social arrangements used in the classroom as well as the analysis of children's learning. Further, this constructivist position was the essence of the professional development sessions conducted with approximately 20 teachers which the teacher under consideration partook. As a consequence, the value the teacher places on children and their thinking and her attempts to keep this the focus of her teaching is key to understanding the functioning of the class.

Therefore, with this in mind, an investigation of the social processes and the interaction relies on a theoretical orientation drawn from social psychology and sociology found in the work of interactionists such as Blumer (1969) and Bauersfeld, Krummheuer, and Voigt (c.f., Cobb & Bauersfeld, 1995). From this perspective, the meaning an individual holds for mathematics is linked to the context in which their actions arise. Teachers and students are seen to mutually establish mathematical interpretations and understandings that form the basis for their communication about mathematics. This process of mutual orienting is thought to influence subjective construction and to also reflect the manner in which common meanings essential for communication are established by the participants. Because it is assumed that the nature of the interaction influences constructive processes which occur in the minds of children, this investigation is not a look at learning as in the qualitative study of Wood (1995) or the quantitative analysis of Wood and Sellers (in press). The intent, instead, is to analyze the nature of the interaction and the teacher's activity within this context.

METHOD, DATA RESOURCE, AND ANALYSIS

The methodology used in the analysis follows those well-established by the interpretist research tradition and is from a qualitative research paradigm. In this case, analysis consists of both participant observer in the classroom and a detached researcher examining videotape records.

The analysis of teaching is situated in a data resource which consists of 50 videotaped lessons collected over a period of one and a half years. These mathematics lessons are approximately 45 minutes in length, consist of pairs of children working together to solve problematic tasks, followed by class discussions of their solutions. This data is part of a larger set of data (250 lessons) collected in several other classrooms for the purpose of analyzing teaching as related to children's mathematical learning. Each lesson is logged for analysis. These logs represent detailed records of the lesson. That is, they capture the events and discourse that occurred during the lesson to allow for the initial line-by-line analysis of the data.

The analysis consisted of examining the structure or form of each lesson. Commonalties were derived in order to identify the consistent and reliable structure or 'typical' lesson. Next, the nature of the interaction for each class discussion was further analyzed using analytic procedures derived from Erickson (1986). This was first accomplished by using a coding scheme developed for the analysis.

The coding categories were grounded in the results from previous analyses of the classroom mentioned in Wood et. al, (1993). Categories were developed to analyze the teachers' discourse with regard to: the norm statements made for students' participation in the discussion, and, the questions and statements made during the discussions. Through a process of toing and froing between analysis of the empirical data and discussion as a research team[2] the categories used in coding the data in the line-by-line analysis were altered and refined.

These were further substantiated by examining studies found in the long tradition of psychological research from an experimental paradigm on the cognitive, affective, and motivational aspects involved in student learning in which the goals involved conceptual understanding. Further, the work of those such as Ainley (1988) on teacher questioning in class discussions, and processes involved in argumentation as in Antaki (1994) were taken into consideration in refining the categories.

Although the coding scheme took nearly three months to develop, it helped to avoid the problem of being "buried in data" and allowed us to move through the videotapes of the lessons quickly and effectively. From this analysis, interaction patterns were identified which represented the consistent reliable forms of social interaction that occurred during the class discussions. These forms of interaction were further analyzed using microanalytic interpretive procedures following Voigt (1990).

For the purpose of this paper, the lessons selected from this analysis were of two origins. One set of lessons consisted of those episodes in which disagreement and/or confusion arose and occurred during the second half of the school year. These were used in the analysis of teacher's discourse that occurred in the discussions in which mathematical argumentation occurred. The other set consisted of the lessons that occurred in the first four weeks of school and twice monthly thereafter. These were used to identified the manner in which the teacher initiated and established the norms that underlie the interaction that occurs during disagreements.

---

[2] I am deeply grateful to Tammy Turner-Vorbeck and William Walker III who painstakingly worked with me to develop the coding categories and collaborated in the analysis of the data.

NATURE OF TEACHING

Limitations of space do not allow for the presentation of dialogue from episodes used to exemplify the process by which the teacher maintains the discussion such that it emerges as a form of mathematical *argumentation*. Space limitations also do not allow the presentation of illustrations of the teacher's discourse as she sustained children in their attempts to participate.

The results from the analysis of the discussion reveal one feature of teaching is the high priority the teacher places on the children, themselves, providing reasons for their personal thinking. It is not enough in this class for pupils to just describe their thinking, instead they must be prepared to answer questions which might arise from the others about their reasoning. The results reveal the intricate manner by which the teacher handles the tension between questioning which enables children to give reasons, and yet is not perceived as taking precedence over their thinking.

However, what distinguishes this teaching from all other forms of instruction is the central importance placed on the role of the listeners as active participants in the discussion by the teacher. It is through the listeners' questioning of explanations that challenges arise. Additionally, the manner by which the teacher sustains the interaction until children reach a resolution reveals not only a form of mathematical argumentation, but the delicate balance of questioning the teacher uses to accomplish this.

ESTABLISHING NORMS

The manner in which the teacher initiated with her students the norms that would support situations in which the children would resolve their disagreements for themselves was revealed in the discussion on the first day of school. The teacher's sensitivity to the inherent difficulties that might arise for the children during instances of disagreement are apparent. During this discussion, the teacher addresses an issue which often is not distinguished by young children. That is, the difference between the meaning of a disagreement that is personal and one which arises from differences in ways of thinking mathematically.

In the weeks that followed, as the teacher interacted with the children, she intentionally replicated the manner in which she expected them to participate during discussion. When the children were explaining, she frequently asked questions which required pupils to provide reasons for their thinking. Initially, the reasons given by the children were vague or related to their physical actions. Over time, children began to provide reasons that included mathematical logic or inferencing.

Interestingly, the teacher's normative comments during this time were most frequently addressed to the children as listeners. She consistently emphasized their responsibilities during the discussion. This was encapsulated in the key words 'active listener'; the meaning of which also changed in degree over time. Initially, it simply meant *'listen and pay attention'*, but it quickly evolved through several negotiations to mean something far more complex; *"listen, see if your way is the same or different. If different, decide whether you agree or not. If you disagree, tell why you do."* This negotiated responsibility of the listener was crucial to the formulation of the ways in which disagreements were resolved. It provided the avenue for students to challenge one another's thinking, for the explainer to offer justification and in some cases proof for their reasoning, and for listeners to respond, continuing until the class determined the disagreement or confusion was resolved to their satisfaction. As might be expected, this did not always occur during a single lesson, and conceptual understanding of those aspects of mathematics children found difficult continued to arise as a topic for discussion. As the children continued to participate these negotiated meanings became the common ground from which the class was able to progress their mathematical thinking. Thus, this way of teaching not only accommodates individual learning, but is reflective of Dewey's pedagogical concern for community and students' participation in the process of the negotiation of 'collectively-held' knowledge.

At first blush the events described in this paper seem to be about teaching mathematics in a class filled with small children. But at another level, it is about fully realizing the importance of the teacher in creating highly complex networks of social norms which are essential to the nature of the interaction, the form of which ultimately determines how and what mathematics is learned in school.

REFERENCES

Ainley, J. (1988). Perceptions of teachers' questioning styles, In Proceedings of the Twelfth International Conference on the Psychology of Mathematics Education(pp. 92-99). Vesprim, Hungary: PME.

Antaki, C. (1994). Explaining and arguing. London: SAGE

Blumer, H. (1969). Symbolic interactionism. Englewood Cliffs, NJ: Prentice Hall.

Cobb, P., & Bauersfeld, H. (1995). The emergence of mathematical meaning. Hillsdale, NJ: Lawerence Erlbaum Associates.

Confrey, J. (1994). A theory of intellectual development (Part 1). For the Learning of Mathematics, 14, 2-8.

Devlin, K. (1994) Mathematics the science of patterns. New York: Scientific American Library.

Erickson, F. (1986). Qualitative methods in research on teaching. In M.C. Wittrock (Ed.), Handbook of research on teaching (3rd ed.) (pp. 119-161). New York: Macmillan.

Newman, D., Griffin, P., & Cole, M., (1989). The construction zone: Working for cognitive change in school. New York, NY: Cambridge University Press.

Voigt, J. (1990). The microethnographical investigation of the interactive constitution of mathematical meaning. Proceedings of the 2nd Bratislava International Symposium on Mathematics Education(pp. 120-143). Bratislava, Czechoslavakia: Bratislava University Press.

Wood, T. (1995). Events in learning mathematics: Insights from research in classrooms. Educational Studies in Mathematics, 29, 1-22.

Wood, T., & Sellers, P. (in press). Deepening the analysis: A longitudinal assessment of a problem-centered mathematics program. Journal for Research in Mathematics Education.

Wood, T., Cobb, P., Yackel, E., & Dillon, D. (1993). Rethinking Elementary Mathematics Education: Insights and Issues, Journal of Research in Mathematics Education Monograph, 6. Reston, VA: National Council of Teachers of Mathematics.